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Abstract. We present a consistent formulation of quantum game theory that accommodates all possible strategies in Hilbert space. The physical content of the quantum strategy is revealed as a family of classical games representing altruistic game play supplemented by quantum interferences. Crucial role of the entanglement in quantum strategy is illustrated by an example of quantum game representing the Bell's experiment.

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INTRODUCTION

The quantum game theory [1, 2, 3, 4, 5] has two aspects. From one side, it is an extension of conventional game theory with Hilbert space vectors and operators. From the other side, it is an attempt to reformulate the description of quantum information processing with the concept of payoff maximization.

In conventional game theory, strategies of players are represented by real-valued vectors, and payoffs by real-valued matrices with no further specifications. In quantum game theory, they are replaced by complex *unitary* vectors and *Hemitian* matrices. It appears that the criterion of mathematical beauty alone favors the latter over the former. Since the space of classical strategies forms a subset of the entire quantum strategy space, it is quite natural to regard the game theory formulated on Hilbert space as a logical extension of classical game theory. It is tempting to imagine that, in search of natural extension, the quantum game theory could have eventually been found irrespective to the discovery of quantum mechanics itself. A crucial questions then arise: What is the *physical content* of quantum strategies? Which part of a quantum strategy is classically interpretable and which part purely quantum? Answers to these questions should also supply a key to understand the mystery surrounding the "quantum resolution" of games with classical dilemmas [6, 7].

Obviously, the answers to these questions are to be obtained only through a consistent formulation of game strategies on Hilbert space. When that is achieved, it can be used as a springboard to deal with the second aspect of the quantum game theory; namely, quantum games played with microscopic objects in states with full quantum superposition and entanglement. In quantum information theory, concept of efficiency occasionally arises. That would supply the payoff function once we are able to identify "game players" in the information processing. It should then become possible to reformulate the problem with the language of quantum games.

In this note, we formulate quantum strategies for classical games with *diagonal payoff matrices*, and clarify the classical and quantum contents of the resulting payoff function.

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We will discover two striking features in the results; the existence of a third party, and the mixture of altruistic strategies. We also sketch the game theoretic formulation of quantum information processing through an example of Bell's experiment. We naturally recover the Tsirelson's limit.

GAME STRATEGY AND PAYOFF ON HILBERT SPACE

We start by considering n-dimensional Hilbert spaces $\mathscr{H}_A$ and $\mathscr{H}_B$ in which the strategies of the two players A and B are represented by vectors $|\alpha\rangle_A \in \mathscr{H}_A$ and $|\beta\rangle_B \in \mathscr{H}_B$. The space of *joint strategies* of the game is given by the direct product $\mathscr{H} = \mathscr{H}_A \times \mathscr{H}_B$. A vector in $\mathscr{H}$ representing a joint strategy of the two players can be written [8] as

$$|\alpha,\beta;\gamma\rangle = J(\gamma)\,|\alpha\rangle_A\,|\beta\rangle_B\,, \tag{1}$$

where the unitary operator $J(\gamma)$ provides quantum correlation (*e.g.*, entanglement) for the separable states $|\alpha\rangle_A\,|\beta\rangle_B$. The two-body operator $J(\gamma)$ is independent of the players' choice and is determined by a third party, which can be regarded as a *coordinator* of the game.

Once the joint strategy is specified with $J(\gamma)$, the players are to receive the payoffs, which are given by the expectation values of Hermitian operators A and B:

$$\begin{aligned} \Pi_A(\alpha,\beta;\gamma) &= \langle\alpha,\beta;\gamma|A|\alpha,\beta;\gamma\rangle\,, \\ \Pi_B(\alpha,\beta;\gamma) &= \langle\alpha,\beta;\gamma|B|\alpha,\beta;\gamma\rangle\,. \end{aligned} \tag{2}$$

Both players try to optimize their strategy to gain the maximal payoff, and the result is the quantum version of the Nash equilibrium, where we have $(\alpha,\beta) = (\alpha^\star,\beta^\star)$ in the strategy space, at which point the payoffs separately attain the maxima as

$$\delta_\alpha \Pi_A(\alpha,\beta^\star;\gamma)\big|_{\alpha^\star} = 0, \quad \delta_\beta \Pi_B(\alpha^\star,\beta;\gamma)\big|_{\beta^\star} = 0, \tag{3}$$

under arbitrary variations in α and β. We express the individual strategies in terms of orthonormal basis strategies $\{|i\rangle\}$, $i = 1,\ldots,n$ which we regard as common to A and B.

$$|\alpha\rangle_A = \sum_i \alpha_i\,|i\rangle_A\,, \quad |\beta\rangle_B = \sum_i \beta_i\,|i\rangle_B\,, \tag{4}$$

with complex numbers α_i, β_i normalized as $\sum_i |\alpha_i|^2 = \sum_i |\beta_i|^2 = 1$. We introduce the swap operator S by

$$S\,|i,j\rangle = |j,i\rangle \tag{5}$$

for the states $|i,j\rangle = |i\rangle_A\,|j\rangle_B$, and then $S\,|\alpha,\beta\rangle = |\beta,\alpha\rangle$ for general separable states $|\alpha,\beta\rangle = |\alpha\rangle_A\,|\beta\rangle_B$ results. We further introduce operators C and T by

$$C\,|i,j\rangle = |\bar{i},\bar{j}\rangle\,, \quad T\,|i,j\rangle = |\bar{j},\bar{i}\rangle\,, \tag{6}$$

where the bar represents the complimentary choice; $\bar{i} = (n-1) - i$. The operator C is the simultaneous renaming (conversion) of strategy for two players, and T is the combination $T = CS$. These operators $\{S, C, T\}$ commute among themselves and satisfy

$$\begin{aligned} S^2 = C^2 = T^2 = I, \\ T = SC, S = CT, C = TS, \end{aligned} \tag{7}$$

where I is the identity operator. They form the dihedral group D_2.

By defining the *correlated payoff operators*

$$\mathscr{A}(\gamma) = J^\dagger(\gamma) A J(\gamma), \quad \mathscr{B}(\gamma) = J^\dagger(\gamma) B J(\gamma), \tag{8}$$

we have $\Pi_A(\alpha, \beta; \gamma) = \langle \alpha, \beta | \mathscr{A}(\gamma) | \alpha, \beta \rangle$. We consider diagonal payoff matrices whose elements are given by

$$\begin{aligned} \langle i', j' | A | i, j \rangle &= A_{ij} \delta_{i'i} \delta_{j'j}, \\ \langle i', j' | B | i, j \rangle &= B_{ij} \delta_{i'i} \delta_{j'j}. \end{aligned} \tag{9}$$

Observe that we have

$$\begin{aligned} \Pi_A(\alpha, \beta; 0) = \sum_{i,j} x_i A_{ij} y_j, \\ \Pi_B(\alpha, \beta; 0) = \sum_{i,j} x_i B_{ij} y_j, \end{aligned} \tag{10}$$

where $x_i = |\alpha_i|^2$ and $y_j = |\beta_j|^2$ are the probability of choosing the strategies $|i\rangle_A$ and $|j\rangle_B$ respectively. This means that, at $\gamma = 0$, our quantum game reduces to the classical game with the payoff matrix A_{ij} under mixed strategies.

ALTRUISTIC CONTENTS AND QUANTUM INTERFERENCES IN QUANTUM GAMES

Let us now restrict ourselves to two strategy games $n = 2$. The unitary operator $J(\gamma)$ then admits the form,

$$J(\gamma) = e^{i\gamma_1 S/2} e^{i\gamma_2 T/2}, \tag{11}$$

where $\gamma = (\gamma_1, \gamma_2)$ are real parameters. Note that, on account of the relation $S + T - C = I$ valid for $n = 2$, only two operators are independent in the set $\{S, C, T\}$. The correlated payoff operator $A(\gamma)$ is split into two terms

$$\mathscr{A}(\gamma) = \mathscr{A}^{\rm pc}(\gamma) + \mathscr{A}^{\rm in}(\gamma) \tag{12}$$

where $\mathscr{A}^{\rm pc}$ is the "pseudo classical" term and $\mathscr{A}^{\rm in}$ is the "interference" term given, respectively, by

$$\begin{aligned} \mathscr{A}^{\rm pc}(\gamma) &= \cos^2 \frac{\gamma_1}{2} A + (\cos^2 \frac{\gamma_2}{2} - \cos^2 \frac{\gamma_1}{2}) SAS + \sin^2 \frac{\gamma_2}{2} CAC, \\ \mathscr{A}^{\rm in}(\gamma) &= \frac{i}{2} \sin \gamma_1 (AS - SA) + \frac{i}{2} \sin \gamma_2 (AT - TA). \end{aligned} \tag{13}$$

Correspondingly, the full payoff is also split into two contributions from $\mathscr{A}^{\rm pc}$ and $\mathscr{A}^{\rm in}$ as $\Pi_A = \Pi_A^{\rm pc} + \Pi_A^{\rm in}$. To evaluate the payoff, we may choose both α_0 and β_0 to be real without loss of generality, and adopt the notaions $(\alpha_0, \alpha_1) = (a_0, a_1 e^{i\xi})$ and $(\beta_0, \beta_1) = (b_0, b_1 e^{i\chi})$. The outcome is

$$\Pi_A^{\rm pc}(\alpha, \beta; \gamma) = \sum_{i,j} a_i^2 b_j^2 \mathscr{A}_{ij}^{\rm pc}(\gamma), \tag{14}$$

$$\Pi_A^{\rm in}(\alpha, \beta; \gamma) = -a_0 a_1 b_0 b_1 [G_+(\gamma)\sin(\xi+\chi) + G_-(\gamma)\sin(\xi-\chi)],$$

with $\mathscr{A}_{ij}^{\rm pc}(\gamma) = \langle i, j | \mathscr{A}^{\rm pc}(\gamma) | i, j \rangle$ and

$$\begin{aligned} G_+(\gamma) &= (A_{00} - A_{11}) \sin\gamma_2, \\ G_-(\gamma) &= (A_{01} - A_{10}) \sin\gamma_1. \end{aligned} \tag{15}$$

A completely parallel expressions are obtained for the payoff matrix $B(\gamma)$ and the payoff $\Pi_B(\alpha, \beta; \gamma)$ for the player B.

Above split of the payoff shows that the quantum game consists of two ingredients. The first is the pseudo classical ingredient associated with $\mathscr{A}^{\rm pc}(\gamma)$, whose form indicates that we are, in effect, simultaneously playing three different classical games, *i.e.,* the original classical game A, and two types of "converted" games, specified by diagonal matrices SAS and CAC with the mixture specified by given γ_1 and γ_2. Regarding γ as tunable parameters, we see that the quantum game contains a *family* of classical games that includes the original game. The second ingredient of the quantum game is the purely quantum component $\mathscr{A}^{\rm in}(\gamma)$, which occurs only when both of the two players adopt quantum strategies with $a_0 a_1 b_0 b_1 \neq 0$ and non-vanishing phases ξ and χ. The structure of $\Pi_A^{\rm in}$ suggests that this interference term cannot be simulated by a classical game and hence represents the *bona fide* quantum aspect.

We further look into the pseudo classical family to uncover its physical content. To that end, we assume that one of the coordinator's parameters, γ_2 is zero. We have

$$\begin{aligned} \mathscr{A}^{\rm pc}(\gamma_1) &= \cos^2\frac{\gamma_1}{2} A + \sin^2\frac{\gamma_1}{2} SAS \\ \mathscr{B}^{\rm pc}(\gamma_1) &= \cos^2\frac{\gamma_1}{2} B + \sin^2\frac{\gamma_1}{2} SBS \end{aligned} \tag{16}$$

The meaning of these payoff matrices becomes evident by considering a *symmetric game*, which is defined by requiring that the payoffs are symmetric for two players, namely $\Pi_A(\alpha, \beta; \gamma) = \Pi_B(\beta, \alpha; \gamma)$. The game appears identical to both players A and B. In this sense, a symmetric game is *fair* to both parties. It is easy to see that the condition of symmetry translates into the requirement $B = SAS$. We then have, for a symmetric game,

$$\begin{aligned} \mathscr{A}^{\rm pc}(\gamma_1) &= \cos^2\frac{\gamma_1}{2} A + \sin^2\frac{\gamma_1}{2} B \\ \mathscr{B}^{\rm pc}(\gamma_1) &= \cos^2\frac{\gamma_1}{2} B + \sin^2\frac{\gamma_1}{2} A. \end{aligned} \tag{17}$$

This means that the pseudo classical game specified by modified rule $\mathscr{A}(\gamma_1)$ and $\mathscr{B}(\gamma_1)$ can be interpreted as a game played with the mixture of *altruism*, or players' taking into account of other party's interest along with their own self-interest [9, 10]. The degree of mixture of altruism is controlled by the correlation parameter γ_1. It is a well known fact that altruistic behavior is widespread among primates that lead social life. It is also well known that the introduction of altruism "solves" such long-standing problems as prisoner's dilemma, to which attempts for solution within conventional game theories based solely on narrow egoistic self-interest has been notoriously difficult [11, 12].

If we fix the first correlation parameter to be $\gamma_1 = \pi/2$, and assume *T-symmetric* game $B = TAT$, we arrive at a parallel relation to (17), thereby proving the fact that pseudo classical family is essentially made up of classical games with altruistic modification specified by the coordinator's parameter γ.

For detailed solutions of Nash equilibria with exhaustive classification according to the relative value of the payoff parameters, readers are referred to [8, 13, 5].

BELL EXPERIMENT AS A QUANTUM GAME

What we have done up to now amounts to "quantizing" classical games. With the advent of nanotechnology, however, it is now possible to actually set up a *game with quantum particles* as a laboratory experiment that has no classical analogue. For such quantum games, we have to allow arbitrary Hermitian payoff operators A and B, removing the restriction to diagonal ones, (9). Without the diagonal condition, however, it turns out that the parametrization of Hilbert space $\mathscr{H}_A \times \mathscr{H}_B$ with the correlation operator (11) is not completely valid. (It leaves certain relative phases between basis states fixed, which, for the case of diagonal payoff operators, does no harm.) Instead, we resort to the scheme devised by Cheon, Ichikawa and Tsutsui [14] that utilizes Schmidt decomposition

$$|\Psi(\alpha,\beta;\eta)\rangle = U(\alpha) \otimes U(\beta)\, |\Phi(\eta)\rangle\,, \tag{18}$$

with "initial" correlated state

$$|\Phi(\eta)\rangle = \cos\frac{\eta_1}{2}\,|00\rangle + e^{i\eta_2}\sin\frac{\eta_1}{2}\,|11\rangle\,, \tag{19}$$

and individual $SU(2)$ rotations $U(\alpha)$ and $U(\beta)$ that are controlled respectively by player A and B. For definiteness we write

$$U(\alpha) = \begin{pmatrix} \cos\frac{\theta_\alpha}{2} & -e^{-i\varphi_\alpha}\sin\frac{\theta_\alpha}{2} \\ e^{i\varphi_\alpha}\sin\frac{\theta_\alpha}{2} & \cos\frac{\theta_\alpha}{2} \end{pmatrix}. \tag{20}$$

The Schmidt state $|\Psi(\alpha,\beta;\eta)\rangle$ covers *entire* Hilbert space $\mathscr{H}_A \times \mathscr{H}_B$. Note that the coordinator's parameters η_1 and η_2 have definite meaning as the measure of size and phase of *two-particle entanglement*.

As an example of such quantum game, let us consider payoff operators

$$A = B = \sqrt{2}\,(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)\,. \tag{21}$$

This is nothing other than the measurement operator for Bell's experiment, in which the projection of two spin 1/2 particles specified by the state (18) are measured separately.

Here we identify $|0\rangle$ and $|1\rangle$ as "up" and "down" states of spin 1/2 along z axis, namely

$$\sigma_z|0\rangle = |0\rangle, \quad \sigma_z|1\rangle = -|1\rangle. \tag{22}$$

The spin projection of the first particle is measured either along positive x axis (whose value, we call P_1) or along positive z axis (whose value is P_2) with random alternation. The spin projection of the second particle is measured either along the line 45 degrees between positive x and z axes (Q_1), or along the line 45 degrees between negative x and positive z axes (Q_2), again in random alternation. Suppose that both players are interested in maximizing the quantity

$$\Pi \equiv P_1 Q_1 - P_2 Q_2. \tag{23}$$

We can easily show that Π is given by the common payoff to A and B given by

$$\Pi = \Pi_A(\alpha,\beta,\eta) = \Pi_B(\alpha,\beta,\eta) = \langle\Psi(\alpha,\beta;\eta)|A|\Psi(\alpha,\beta;\eta)\rangle. \tag{24}$$

The game now becomes a one of quantum coordination between players A and B who both try to increase the common payoff Π by respectively controlling the directions of spins with $U(\alpha)$ and $U(\beta)$. Considering the relation

$$\langle\Psi(\alpha,\beta;\eta)|A|\Psi(\alpha,\beta;\eta)\rangle = \langle\Phi(\eta)|\left(U^\dagger(\alpha)\otimes U^\dagger(\beta) A U(\alpha)\otimes U(\beta)\right)|\Phi(\eta)\rangle. \tag{25}$$

we can also restate the game as two players, receiving the correlated two particle state $|\Phi(\eta)\rangle$, trying to maximize the common payoff Π by rotating the directions of spin projection measurement: The player A applies a common rotation $U(\alpha)$ to the directions P_1 and P_2, and the player B applies another common $U(\beta)$ to Q_1 and Q_2.

For a fixed set of entanglement parameters (η_1, η_2), a straightforward calculation yields the Nash equilibrium that is specified by

$$\theta_\alpha^\star = \theta_\beta^\star = arbitrary, \quad \varphi_\alpha^\star = \varphi_\beta^\star = 0, \tag{26}$$

for which, the Nash payoff is given by

$$\Pi^\star(\eta_1,\eta_2) = \sqrt{2}(1+\sin\eta_1\cos\eta_2). \tag{27}$$

For particles with no entanglement, $\eta_1 = 0$, we obtain $\Pi^\star = \sqrt{2}$, which is the known maximum for two uncorrelated spins. For particles with maximum entanglement, $\eta_1 = \pi/2$ and phase $\eta_2 = 0$, we obtain the payoff $\Pi^\star = \sqrt{8}$, which is exactly on the Tsirelson's bound [15].

This reformulation of Bell's experiment should give a hint for the way toward more general game-theoretic reformulation of quantum information processing.

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