

1999年度版 基礎数学ワークブック 番外編 No.2 「ベクトル解析入門」

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高知工科大学

基礎数学ワークブック

(1999年度版)

番外編 *No.2*

「ベクトル解析入門」

解答

< 2 ページ. 平面上の道のり 2 >

問 1 の解答

$$\ell = \int_0^{\theta} \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt = \int_0^{\theta} r dt = \theta r$$

問 2 の解答

$$\ell = \int_a^b \sqrt{1 + (f'(t))^2} dt$$

< 3 ページ. 線と面 >

問 1 の解答

$$\begin{aligned} S &= \int_0^3 (-x^2 + 3x) dx = \left[-\frac{x^3}{3} + \frac{3}{2}x^2 \right]_0^3 \\ &= -9 + \frac{27}{2} \\ &= \frac{9}{2} \end{aligned}$$

問 2 の解答

$$S = \pi R^2 \quad , \quad \ell(r) = 2\pi r$$

問 3 の解答

$$S = \int_0^R \theta r dr = \frac{\theta}{2} R^2 \quad , \quad \ell(r) = \theta r$$

< 4 ページ. 面と立体 >

問の解答

$$\begin{aligned} V &= \int_{-r}^r \pi \left(\sqrt{r^2 - x^2} \right)^2 dx = \int_{-r}^r \pi (r^2 - x^2) dx \\ &= \left[\pi r^2 x - \frac{\pi}{3} x^3 \right]_{-r}^r \\ &= \pi r^3 - \frac{\pi}{3} r^3 - \left(-\pi r^3 + \frac{\pi}{3} r^3 \right) \\ &= \frac{2}{3} \pi r^3 - \left(-\frac{2}{3} \pi r^3 \right) \\ &= \frac{4}{3} \pi r^3 \end{aligned}$$

< 5 ページ. 質量と重心 1 >

問の解答

$$g = \frac{1}{M} \{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n\}$$

< 7 ページ. 質量と重心 3 >

問の解答

$$\begin{aligned} M &= \int_0^2 f(x)dx = \int_0^2 (-x^2 + 2x)dx \\ &= \left[-\frac{x^3}{3} + x^2 \right]_0^2 \\ &= -\frac{8}{3} + 4 \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} g &= \frac{1}{M} \int_0^2 x f(x)dx = \frac{1}{\frac{4}{3}} \int_0^2 (-x^3 + 2x^2)dx \\ &= \frac{3}{4} \left[-\frac{x^4}{4} + \frac{2}{3}x^3 \right]_0^2 \\ &= \frac{3}{4} \left\{ -4 + \frac{16}{3} \right\} \\ &= \frac{3}{4} \times \frac{4}{3} \\ &= 1 \end{aligned}$$

< 8 ページ. 質量と重心 4 >

問の解答

$$\begin{aligned}g_Y &= \frac{1}{M} \iint_D yf(x, y) dx dy = \frac{1}{3} \iint_{D_1} y dx dy + \frac{1}{3} \iint_{D_2} y dx dy \\&= \frac{1}{3} \int_0^1 \left\{ \int_0^{2x} y dy \right\} dx + \frac{1}{3} \int_1^3 \left\{ \int_0^{-x+3} y dy \right\} dx \\&= \frac{1}{3} \int_0^1 \left\{ \left[\frac{y^2}{2} \right]_{y=0}^{y=2x} \right\} dx + \frac{1}{3} \int_1^3 \left\{ \left[\frac{y^2}{2} \right]_{y=0}^{y=-x+3} \right\} dx \\&= \frac{1}{3} \int_0^1 2x^2 dx + \frac{1}{3} \int_1^3 \frac{(-x+3)^2}{2} dx \\&= \frac{1}{3} \left[\frac{2}{3} x^3 \right]_0^1 + \frac{1}{3} \left[-\frac{(-x+3)^3}{6} \right]_1^3 \\&= \frac{2}{9} + \frac{1}{3} \left(0 + \frac{8}{6} \right) = \frac{2}{9} + \frac{4}{9} \\&= \frac{2}{3}\end{aligned}$$

< 9 ページ. 回転体の表面積 1 >

問の解答

$$S = \pi m \sqrt{1+m^2} b^2 - \pi m \sqrt{1+m^2} a^2$$

< 10 ページ. 回転体の表面積 2 >

問の解答

$$S = \int_1^3 6\pi dx = 12\pi$$

< 11 ページ. 回転体の表面積 3 >

問の解答

$$S = \int_{-R}^R 2\pi y \sqrt{1 + (y')^2} dx = \int_{-R}^R 2\pi R dx = 4\pi R^2$$

< 13 ページ. 平面上の運動 2 >

問 1 の解答

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \frac{d\mathbf{v}}{dt} = \left(\frac{d}{dt}(-6\pi \sin(2\pi t)), \frac{d}{dt}(6\pi \cos(2\pi t)) \right) \\ &= (-12\pi^2 \cos(2\pi t), -12\pi^2 \sin(2\pi t))\end{aligned}$$

$$\left| \frac{d^2\mathbf{r}}{dt^2} \right| = \sqrt{(-12\pi^2 \cos(2\pi t))^2 + (-12\pi^2 \sin(2\pi t))^2} = 12\pi^2$$

問 2 の解答

$$S = \int_0^{\frac{1}{4}} \left| \frac{d\mathbf{r}}{dt} \right| dt = \int_0^{\frac{1}{4}} 6\pi dt = 6\pi \times \frac{1}{4} = \frac{3}{2}\pi$$

< 14 ページ. 平面上の線積分 1 >

問 1 の解答

$$\int_C xy dt = \int_{-1}^1 x(t)y(t) dt = \int_{-1}^1 t \times t^2 dt = \left[\frac{1}{4} t^4 \right]_{-1}^1 = 0$$

問 2 の解答

$$\begin{aligned} \int_C (x + y) dt &= \int_0^{\frac{\pi}{2}} (3 \cos t + 3 \sin t) dt = \left[3 \sin t - 3 \cos t \right]_0^{\frac{\pi}{2}} \\ &= 3 \sin \left(\frac{\pi}{2} \right) - 3 \cos \left(\frac{\pi}{2} \right) - 3 \sin 0 + 3 \cos 0 \\ &= 3 - 0 - 0 + 3 \\ &= 6 \end{aligned}$$

< 16 ページ. 平面上の線積分 3 >

問の解答

$$\begin{aligned}(1) \quad & \int_C (x^2 + y^2) ds \\ &= \int_0^{2\pi} (r^2 \cos^2 t + r^2 \sin^2 t) \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= \int_0^{2\pi} r^2 \times r dt \\ &= 2\pi r^3\end{aligned}$$

$$\begin{aligned}(2) \quad & \int_C (x + y) ds \\ &= \int_0^{2\pi} (r \cos t + r \sin t) \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt \\ &= \int_0^{2\pi} (r \cos t + r \sin t) r dt \\ &= r^2 \left[\sin t - \cos t \right]_0^{2\pi} \\ &= r^2 \left\{ \sin(2\pi) - \cos(2\pi) - \sin 0 + \cos 0 \right\} \\ &= 0\end{aligned}$$

< 17 ページ. 平面上の線積分 4 >

問の解答

$$\begin{aligned}(1) \int_C (x+y)dx &= \int_0^1 (x(t) + y(t)) \frac{dx}{dt} dt \\ &= \int_0^1 (t + \sqrt{t}) dt \\ &= \left[\frac{t^2}{2} + \frac{2}{3} t^{\frac{3}{2}} \right]_0^1 \\ &= \frac{1}{2} + \frac{2}{3} \\ &= \frac{7}{6}\end{aligned}$$

$$\begin{aligned}(2) \int_C (x+y)dy &= \int_0^1 (x(t) + y(t)) \frac{dy}{dt} dt \\ &= \int_0^1 (t + \sqrt{t}) \frac{1}{2\sqrt{t}} dt = \int_0^1 \left(\frac{1}{2} \sqrt{t} + \frac{1}{2} \right) dt \\ &= \left[\frac{1}{3} t^{\frac{3}{2}} + \frac{1}{2} t \right]_0^1 \\ &= \frac{1}{3} + \frac{1}{2} \\ &= \frac{5}{6}\end{aligned}$$

< 18 ページ. 平面上の線積分 5 >

問の解答

$$S_2 = \int_a^b \psi(x) dx = - \int_b^a y dx = - \int_C y dx$$

< 19 ページ. 平面上の線積分 6 >

問の解答

$$S = \int_{C_2} xdy - \int_{-C_1} xdy = \int_{C_2} xdy + \int_{C_1} xdy = \int_C xdy$$

< 20 ページ. グリーンの定理 1 >

問の解答

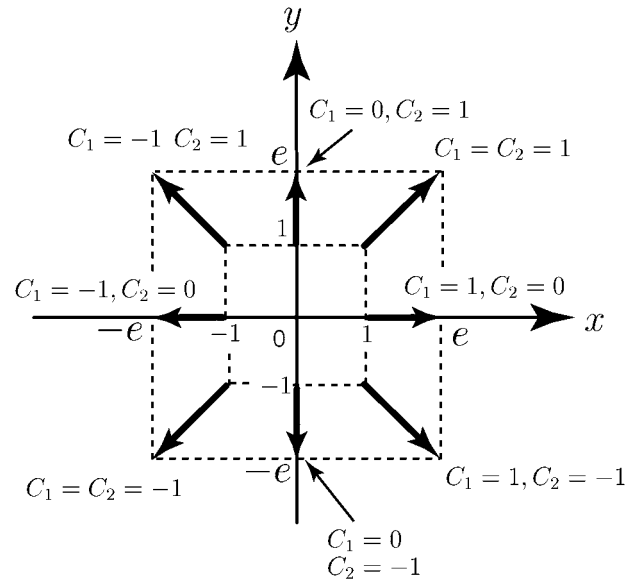
$$\iint_D \frac{\partial f}{\partial y} dx dy = - \int_b^a f(x, \psi(x)) dx - \int_a^b f(x, \varphi(x)) dx = - \int_C f(x, y) dx$$

< 22 ページ. 平面上の流れ 1 >

問の解答

$$\frac{dx}{dt} = x \Rightarrow x(t) = C_1 e^t$$

$$\frac{dy}{dt} = y \Rightarrow y(t) = C_2 e^t$$



< 23 ページ. 平面上の流れ 2 >

問の解答

$$\operatorname{div}(\boldsymbol{v}) = \operatorname{div}(-x, y) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(y) = -1 + 1 = 0$$

< 24 ページ. 平面上の流れ 3 >

問の解答

$$(1) \mathbf{v} = (2x, 2y)$$

$$\operatorname{div}(\mathbf{v}) = 2 + 2 = 4$$

$$\operatorname{rot}(\mathbf{v}) = 0$$

$$(2) \mathbf{v} = (2x - y, 2y + x)$$

$$\operatorname{div}(\mathbf{v}) = 2 + 2 = 4$$

$$\operatorname{rot}(\mathbf{v}) = 1 - (-1) = 2$$

$$(3) \mathbf{v} = (2x + 3y, 4x - 5y)$$

$$\operatorname{div}(\mathbf{v}) = 2 - 5 = -3$$

$$\operatorname{rot}(\mathbf{v}) = 4 - 3 = 1$$

< 25 ページ. 平面上の流れ 4 >

問の解答

$$U(x, y) = -\frac{1}{2}x^2 - \frac{1}{2}y^2 \text{ とおくと}$$

$$\left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y} \right) = \left(-(-x), -(-y) \right) = (x, y) = \boldsymbol{v}$$

より $U(x, y) = -\frac{1}{2}x^2 - \frac{1}{2}y^2$ がポテンシャルである。

< 28 ページ. 平面のベクトル場の線積分 3 >

問の解答

$$\begin{aligned}\iint_D \operatorname{rot}(\mathbf{F}) dx dy &= \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx dy \\ &= \int_C (f_1 dx + f_2 dy) = \int_C \mathbf{F} \cdot d\mathbf{r}\end{aligned}$$

< 29 ページ. 空間のベクトル 1 >

問の解答

$$(1) |\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$(2) |\mathbf{b}| = \sqrt{3^2 + 0^2 + (-1)^2} = \sqrt{10}$$

$$(3) \mathbf{a} \cdot \mathbf{b} = 1 \times 3 + 2 \times 0 + 3 \times (-1) = 0$$

$$(4) \theta = 90^\circ = \frac{\pi}{2}$$

$$\begin{aligned} (5) S &= |\mathbf{a}| \cdot |\mathbf{b}| \cdot \sin \theta \\ &= \sqrt{14} \times \sqrt{10} \times \sin \left(\frac{\pi}{2} \right) \\ &= \sqrt{140} \\ &= 2\sqrt{35} \end{aligned}$$

< 30 ページ. 空間のベクトル 2 >

問の解答

$$(1) \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 10\mathbf{k} = (0, 0, 10)$$

$$(2) \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 12\mathbf{i} - 3\mathbf{j} - 3\mathbf{k} = (12, -3, -3)$$

$$(3) (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (0, 0, 10) \cdot (1, 1, 3) = 30$$

$$(4) (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (12, -3, -3) \cdot (3, 2, 0) = 30$$

< 31 ページ. 空間の運動 >

問の解答

$$\mathbf{v} = (-\sin t, \cos t, 0.2)$$

$$|\mathbf{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (0.2)^2}$$

$$= \sqrt{1 + 0.04}$$

$$= \sqrt{1.04}$$

$$s = \int_0^{2\pi} |\mathbf{v}| dt = \int_0^{2\pi} \sqrt{1.04} dt = 2\pi\sqrt{1.04}$$

< 32 ページ. 空間の線積分 >

問の解答

$$\begin{aligned}(1) \int_C f ds &= \int_C (x(t) + y(t) + z(t)) \left| \frac{dr}{dt} \right| dt \\ &= \int_0^{2\pi} (\cos t + \sin t + 0.2t + 1) \sqrt{1.04} dt \\ &= \sqrt{1.04} \times [\sin t - \cos t + 0.1t^2 + t]_0^{2\pi} \\ &= (0.4\pi^2 + 2\pi) \sqrt{1.04}\end{aligned}$$

$$\begin{aligned}(2) \int_C f dz &= \int_0^{2\pi} (x(t) + y(t) + z(t)) \frac{dz}{dt} dt \\ &= \int_0^{2\pi} (\cos t + \sin t + 0.2t + 1) 0.2 dt \\ &= 0.2 \times [\sin t - \cos t + 0.1t^2 + t]_0^{2\pi} \\ &= 0.2(0.4\pi^2 + 2\pi) \\ &= 0.08\pi^2 + 0.4\pi\end{aligned}$$

< 33 ページ. 空間のベクトル場の線積分 >

問の解答

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \{\cos^2 t + \sin^2 t\}(-\sin t)dt \\ &\quad + \int_0^{2\pi} \frac{\sin t}{\cos t} \times \cos t dt \\ &\quad + \int_0^{2\pi} (0.2t + 1) \times 0.2 dt \\ &= [\cos t]_0^{2\pi} + [-\cos t]_0^{2\pi} + 0.2[0.1t^2 + t]_0^{2\pi} \\ &= 0.2(0.4\pi^2 + 2\pi) \\ &= 0.08\pi^2 + 0.4\pi\end{aligned}$$

< 34 ページ. 平面のパラメータ表示 >

問の解答

$$\frac{\partial \mathbf{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) = (b_1, b_2, b_3) = \mathbf{b}$$

$$\mathbf{a} \times \mathbf{b} = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

< 35 ページ. 球面のパラメーター表示 >

問の解答

$$\begin{aligned} & \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \\ &= (-R \cos v \sin u, R \cos v \cos u, 0) \times (-R \sin v \cos u, -R \sin v \sin u, R \cos v) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -R \cos v \sin u & R \cos v \cos u & 0 \\ -R \sin v \cos u & -R \sin v \sin u & R \cos v \end{vmatrix} \\ &= (R^2 \cos^2 v \cos u) \mathbf{i} + (R^2 \cos^2 v \sin u) \mathbf{j} + (R^2 \cos v \sin v \sin^2 u + R^2 \cos v \sin v \cos^2 u) \mathbf{k} \\ &= R \cos v (R \cos v \cos u, R \cos v \sin u, R \sin v) \\ &= R \cos v \mathbf{r}(u, v) \end{aligned}$$

< 37 ページ. 曲面の面積 2 >

問の解答

$$D = \left\{ (u, v) : 0 \leq u \leq 2\pi, -\frac{\pi}{2} \leq v \leq \frac{\pi}{2} \right\}$$

$$\begin{aligned} S &= \iint_D \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| du dv = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ \int_0^{2\pi} R^2 \cos v du \right\} dv = 2\pi R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos v dv \\ &= 2\pi R^2 \left[\sin v \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi R^2 \end{aligned}$$

< 38 ページ. スカラー場の面積分 >

問の解答

$$\begin{aligned}\int_S 1 dS &= \int_0^{2\pi} \left\{ \int_a^b f(u) \sqrt{1 + (f'(u))^2} du \right\} dv \\ &= \int_a^b \left\{ \int_0^{2\pi} f(u) \sqrt{1 + (f'(u))^2} dv \right\} du \\ &= \int_a^b 2\pi f(u) \sqrt{1 + (f'(u))^2} du \\ &= \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx\end{aligned}$$

< 40 ページ. ベクトル場の表面積分 2 >

問の解答

$$\begin{aligned}\int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1 + \cos(2v)}{2} (a \cos u + b \sin u) + \frac{c}{2} \sin(2v) \right) dv \right\} du \\ &= \int_0^{2\pi} \left\{ (a \cos u + b \sin u) \left[\frac{v}{2} + \frac{1}{4} \sin(2v) \right]_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}} + \frac{c}{2} \left[-\frac{1}{2} \cos(2v) \right]_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}} \right\} du \\ &= \frac{\pi}{2} \int_0^{2\pi} (a \cos u + b \sin u) du = \frac{\pi}{2} \left[a \sin u - b \cos u \right]_0^{2\pi} = 0\end{aligned}$$

< 41 ページ. ベクトル場の表面積分 3 >

問の解答

$$\int_S (0, 0, f_3) \cdot d\mathbf{S} = - \iint_D \begin{vmatrix} 0 & 0 & f_3 \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} dx dy = - \iint_D f_3(x, y, z(x, y)) dx dy$$

< 42 ページ. 体積積分 1 >

問の解答

$$\int_V \varphi dV = \int_0^3 \left\{ \int_0^2 \left\{ \int_0^1 \varphi(x, y, z) dz \right\} dy \right\} dx$$

< 43 ページ. 体積積分 2 >

問の解答

$$\begin{aligned}\int_V \varphi dV &= \int_0^4 \left\{ \int_0^3 \left\{ \int_0^2 (x+y+z) dz \right\} dy \right\} dx \\ &= \int_0^4 \left\{ \int_0^3 \left[(x+y)z + \frac{z^2}{2} \right]_{z=0}^{z=2} dy \right\} dx = \int_0^4 \left\{ \int_0^3 (2x+2y+2) dy \right\} dx \\ &= \int_0^4 \left\{ [2xy + y^2 + 2y]_{y=0}^{y=3} \right\} dx = \int_0^4 (6x+15) dx \\ &= [3x^2 + 15x]_{x=0}^{x=4} \\ &= 108\end{aligned}$$

< 45 ページ. 回転 1 >

問の解答

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{z} = (\omega_1, \omega_2, \omega_3) \times (x, y, z)$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\boldsymbol{i} + (\omega_3 x - \omega_1 z)\boldsymbol{j} + (\omega_1 y - \omega_2 x)\boldsymbol{k}$$

より $v_1 = \omega_2 z - \omega_3 y$, $v_2 = \omega_3 x - \omega_1 z$, $v_3 = \omega_1 y - \omega_2 x$

< 46 ページ. 回転 2 >

問の解答

(1) $\boldsymbol{\omega} = (0, 0, \theta)$

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \theta \\ x & y & z \end{vmatrix} = (-y\theta, x\theta, 0)$$

$$\begin{aligned} \operatorname{rot} \boldsymbol{v} &= \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x\theta), \frac{\partial}{\partial z}(-y\theta) - \frac{\partial}{\partial x}(0), \frac{\partial}{\partial x}(x\theta) - \frac{\partial}{\partial y}(-y\theta) \right) \\ &= (0, 0, 2\theta) = 2\boldsymbol{\omega} \end{aligned}$$

(2) $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = (\omega_2 z - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x)$$

$$\begin{aligned} \operatorname{rot} \boldsymbol{v} &= (\omega_1 - (-\omega_1), \omega_2 - (-\omega_2), \omega_3 - (-\omega_3)) \\ &= (2\omega_1, 2\omega_2, 2\omega_3) = 2\boldsymbol{\omega} \end{aligned}$$

< 47 ページ. 発散 >

問の解答

$$(1) \mathbf{v} = (ax - by, ay + bx, cz)$$

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \frac{\partial}{\partial x}(ax - by) + \frac{\partial}{\partial y}(ay + bx) + \frac{\partial}{\partial z}(cz) \\ &= a + b + c \end{aligned}$$

$$(2) \mathbf{v} = (\omega_2 z - \omega_3 y, \omega_3 x - \omega_1 z, \omega_1 y - \omega_2 x)$$

$$\begin{aligned} \operatorname{div} \mathbf{v} &= \frac{\partial}{\partial x}(\omega_2 z - \omega_3 y) + \frac{\partial}{\partial y}(\omega_3 x - \omega_1 z) + \frac{\partial}{\partial z}(\omega_1 y - \omega_2 x) \\ &= 0 + 0 + 0 = 0 \end{aligned}$$

< 48 ページ. ハミルトンの演算子 >

問の解答

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}, \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \\ &= \frac{\partial^2 f_3}{\partial y \partial x} - \frac{\partial^2 f_2}{\partial z \partial x} + \frac{\partial^2 f_1}{\partial z \partial y} - \frac{\partial^2 f_3}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial y \partial z} \\ &= \left(\frac{\partial^2 f_3}{\partial y \partial x} - \frac{\partial^2 f_3}{\partial x \partial y} \right) + \left(-\frac{\partial^2 f_2}{\partial z \partial x} + \frac{\partial^2 f_2}{\partial x \partial z} \right) + \left(\frac{\partial^2 f_1}{\partial z \partial y} - \frac{\partial^2 f_1}{\partial y \partial z} \right) \\ &= 0 + 0 + 0 = 0\end{aligned}$$

< 49 ページ. ポテンシャル >

問1の解答

(1) $v = (x - b_1, y - b_2, z - b_3)$

$$U(x, y, z) = -\frac{1}{2}(x - b_1)^2 - \frac{1}{2}(y - b_2)^2 - \frac{1}{2}(z - b_3)^2$$

(2) $v = (-x, -y, 2z)$

$$U(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} - z^2$$

問2の解答

$$\varphi(x, y, z) = \int_a^x f_1(t, y, z) dt + \int_b^y f_2(a, t, z) dt + \int_c^z f_3(a, b, t) dt$$

$$\frac{\partial \varphi}{\partial x} = f_1(x, y, z)$$

$$\begin{aligned} \frac{\partial \varphi}{\partial y} &= \int_a^x \frac{\partial f_1}{\partial y}(t, y, z) dt + f_2(a, y, z) \\ &= \int_a^x \frac{\partial f_2}{\partial x}(t, y, z) dt + f_2(a, y, z) \\ &= \left[f_2(t, y, z) \right]_{t=a}^{t=x} + f_2(a, y, z) \\ &= f_2(x, y, z) - f_2(a, y, z) + f_2(a, y, z) \\ &= f_2(x, y, z) \end{aligned}$$

$$\begin{aligned} \frac{\partial \varphi}{\partial z} &= \int_a^x \frac{\partial f_1}{\partial z}(t, y, z) dt + \int_b^y \frac{\partial f_2}{\partial z}(a, t, z) dt + f_3(a, y, z) \\ &= \int_a^x \frac{\partial f_3}{\partial x}(t, y, z) dt + \int_b^y \frac{\partial f_3}{\partial y}(a, t, z) dt + f_3(a, y, z) \\ &= \left[f_3(t, y, z) \right]_{t=a}^{t=x} + \left[f_3(a, t, z) \right]_{t=b}^{t=y} + f_3(a, b, z) \\ &= f_3(x, y, z) - f_3(a, y, z) + f_3(a, y, z) - f_3(a, b, z) + f_3(a, b, z) \\ &= f_3(x, y, z) \end{aligned}$$

よって

$$\begin{aligned} \nabla \varphi(x, y, z) &= \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right) = \left(f_1(x, y, z), f_2(x, y, z), f_3(x, y, z) \right) \\ &= \mathbf{F}(x, y, z) \end{aligned}$$

より

$$\nabla \varphi = \mathbf{F}$$

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問の解答

$$\int_C \nabla \varphi \cdot d\mathbf{r} = \int_S \operatorname{rot}(\nabla \varphi) \cdot d\mathbf{S} = \int_S \nabla \times (\nabla \varphi) \cdot d\mathbf{S} = \int_S \mathbf{0} \cdot d\mathbf{S} = 0$$