A NORMAL FORM THEOREM FOR ELEMENTARY ANALYSIS

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Abstract: The formal system in which the Peano's axioms hold for numbers and there are quantifications over predicate variables is said to be classical analysis. In this system the real numbers are definable by predicators as certain sets of rational numbers and universal and existential statements about real numbers are formalizable. The formal system of elementary analysis is the subsystem of classical analysis which is restricted to the comprehension axioms for only elementary predicators in which no quantifiers over predicate variables are contained. And the ω-consistency of a formal system is a stronger property than the simple consistency of the system. We show that a normal form theorem for the formal system of elementary analysis which implies the ω-consistency of the system is proved by applying transfinite induction up to $\varepsilon_{\varepsilon_1}$.

1 Introduction

In [1], Ikeda gave a normal form theorem for the formal system of Peano arithmetic (PA) that is an extension of Hinata's normal form theorem for PA [5] and Mints' normal form theorem for LK [6]:

Theorem (Ikeda)**.** *Every derivation in PA can be transformed into a strongly irreducible derivation with the same end sequent.*

This theorem is proved by applying transfinite induction up to ε_1 and it implies the ω-consistency of PA. It is known that the ω-consistency of PA can be proved by applying transfinite induction up to ε_1, but not up to an ordinal less than ε_1 ([4], [9]).

And in [8], Shirai showed the following theorem on the ω-consistency of the formal system of elementary analysis (EA) :

Theorem (Shirai)**.** *The ω-consistency of EA can be proved by applying transfinite induction up to $\varepsilon_{\varepsilon_1}$, but not up to an ordinal less than $\varepsilon_{\varepsilon_1}$.*

In this paper, we will show that a normal form theorem for EA which implies the ω-consistency of EA can be proved by applying transfinite induction up to $\varepsilon_{\varepsilon_1}$.

2 Formal system *EA*

We define the formal system EA (cf. [8] , [7]).

2.1 The language:

2.11 Denumerably infinitely many free and bound number variables.

2.12 Denumerably infinitely many free and bound 1-place predicate variables.

2.13 The individual constant 0.

2.14 The logical symbols: $\neg, \wedge, \vee, \forall, \exists$ and λ

2.15 Symbols for n-place calculable arithmetic functions and n-place decidable arithmetic predicates ($n \geq 1$) . Especially " $'$ " is the successor function symbol and " $=$ " is the equality symbol.

2.16 Auxiliary symbols:), (, , , $\to$

2.2 Inductive definitions of terms:

2.21 The symbol 0 is a term.

2.22 Every free number variable is a term.

2.23 If t is a term then so is t'.

2.24 If f is a symbol for an n-place calculable arithmetic function ($n \geq 1$) and $t_1, \cdots, t_n$ are terms, then $f(t_1, \cdots, t_n)$ is also a term.

Terms built up according to 2.21 and 2.23 only are called *numerals*.

2.3 Prime formulas are formulas of the form $R(t_1, \cdots, t_n)$ where R is a symbol for an n-place decidable arithmetic predicate ($n \geq 1$) and $t_1, \cdots, t_n$ are terms.

2.4 Inductive definitions of formulas and predicators:

2.41 Every prime formula is a formula.

2.42 Every free predicate variable is a predicator.

2.43 If P is a predicator and t is a term, then $P(t)$ is a formula.

2.44 If A is a formula, then so is $(\neg A)$.

2.45 If A and B are formulas, then so are $(A \wedge B)$ and $(A \vee B)$.

2.46 If $F(a)$ is a formula and a is a free number variable and x is a bound number variable which does not occur in $F(a)$, then $\forall x F(x)$ and $\exists x F(x)$ are formulas.

2.47 If $F(a)$ is a formula and a is a free number variable and x is a bound number variable which does not occur in $F(a)$, then $\lambda x F(x)$ is a predicator.

2.48 If $F(U)$ is a formula and U is a free predicate variable and X is a bound predicate variable which does not occur in $F(U)$, then $\forall X F(X)$ and $\exists X F(X)$ are formulas.

A predicator is said to be *elementary* if it contains no bound predicate variables.

2.5 Sequents are defined as LK in [2].

2.6 Initial sequents:

2.61 A logical initial sequent is a sequent of the form $D \to D$ where D is an arbitrary formula without logical symbols.

2.62 A mathematical initial sequent is a sequent consisting of prime formulas, which becomes a true sequent with every arbitrary substitution of numerals for possible occurrences of free number variables.

2.63 An equality initial sequent is a sequent of the form $s = t, U(s) \to U(t)$ where U is an arbitrary free predicate variable and s and t are arbitrary terms. It is said to be *inessential* if s and t are the same terms or closed terms with different values.

2.7 Inference rules:

We add the following rules to the inference rules of LK ([2], [11]) :

2.71 For second order $\forall$

$$\forall^2\text{-IA} : \frac{F(P), \Gamma \to \Theta}{\forall X F(X), \Gamma \to \Theta}, \qquad \forall^2\text{-IS} : \frac{\Gamma \to \Theta, F(U)}{\Gamma \to \Theta, \forall X F(X)},$$

where P is an arbitrary elementary predicator and X is a bound predicate variable.
Restrictions on predicate variables: The free predicate variable which is designated by U and is called the eigenvariable of the $\forall^2$-IS must not occur in the lower sequent of the inference rule.

2.72 For second order $\exists$

$$\exists^2\text{-IA} : \frac{F(U), \Gamma \to \Theta}{\exists X F(X), \Gamma \to \Theta}, \qquad \exists^2\text{-IS} : \frac{\Gamma \to \Theta, F(P)}{\Gamma \to \Theta, \exists X F(X)},$$

where P is an arbitrary elementary predicator and X is a bound predicate variable.
Restrictions on predicate variables: The free predicate variable which is designated by U and is called the eigenvariable of the $\exists^2$-IA must not occur in the lower sequent of the inference rule.

2.73 For λ abstraction

$$\lambda\text{-IA} : \frac{F(t), \Gamma \to \Theta}{\lambda x F(x)(t), \Gamma \to \Theta}, \qquad \lambda\text{-IS} : \frac{\Gamma \to \Theta, F(t)}{\Gamma \to \Theta, \lambda x F(x)(t)},$$

where t is an arbitrary term.

2.74 CJ-inference (Complete Induction)

$$\frac{\Gamma \to \Theta, F(0) \qquad F(a), \Gamma \to \Theta, F(a') \qquad F(t), \Gamma \to \Theta}{\Gamma \to \Theta}$$

Restriction on number variables: The free number variable which is designated by a and is called the eigenvariable of the CJ-inference must not occur in Γ, Θ and $F(t)$ of the inference rule.

CJ-inference is said to be *constant normal* if its *induction formula* $F(a)$ contains at least one occurrence of its eigenvariable a and its *induction term* t contains at least one free number variable. $A(0), A(a), A(a')$ and $A(t)$ are called *auxiliary formulas* of the inference rule.

2.8 Derivations are defined as usual.

2.9 EA^- is the system obtained from EA by omitting CJ-inferences.

3 The Normal Form Theorem

3.1 Let π be a derivation with the end sequent S and A a formula in π.

3.11 We say that A belongs to *CJ-ancestors* if a descendant (cf. [11]) of A is an auxiliary formula of a CJ-inference in π.

3.12 A is said to be *explicit* if a descendant of A is in S.

3.2 A cut is said to be *inessential* if the cut formula contains no logical symbols; otherwise essential.

3.3 A variable in a derivation is said to be *redundant* if it occurs in an upper sequent of an inference I and does not occur in the lower sequent of I and is not used as the eigenvariable of I.

3.4 Let π be a derivation in EA. Then a logical inference I in π is said to be *reducible with respect to EA* [EA^-] if one of the auxiliary formulas of I is derivable (refutable) in EA [EA^-] provided that it belongs to the antecedent (succedent) of the sequent in which it occurs.

3.5 Let π be a derivation with the end sequent S in EA. Then π is said to be *strongly irreducible* if it satisfies the following conditions:

3.51 It contains no cuts with the exception of inessential ones.

3.52 It contains no redundant variables.

3.53 It contains no inessential equality initial sequents.

3.54 It contains no CJ-inferences with the exception of constant normal ones.

3.55 Every logical inference, whose principal formula belongs to CJ-ancestors, in it is irreducible with respect to EA^-.

3.56 Every logical inference, whose principal formula is explicit, in it is irreducible with respect to EA.

Theorem 1. *We can transform any derivation in EA into a strongly irreducible one with the same end sequent.*

We shall prove this theorem in Section 5.

3.6 A formal system T is called *ω-consistent* if the following condition is satisfied; for every formula $F(a)$, if $F(n)$ is derivable in T for any numeral n, then $\neg\forall x F(x)$ is not derivable in T. In this definition we may assume that $F(a)$ has no free variables except a.

3.7 Let π be a derivation. We say that a sequent S in π *belongs to the end-place* of π if neither a logical inference nor a CJ-inference occurs below S in π . And we say that an inference I in π *belongs to the boundary* of π or is a *boundary inference* of π if the lower sequent of I belongs to the end-place of π and the upper sequents of I do not belong to the end-place of π.

Corollary 1 (After Ikeda [1]). *EA is ω-consistent.*

Proof. Let $F(a)$ be an arbitrary formula such that it has no free variables except a and $F(n)$ is derivable in *EA* for any numeral n. Then, it suffices to show that $\forall x F(x) \rightarrow$ is not derivable in *EA*.

Assume that $\forall x F(x) \rightarrow$ is derivable in *EA*. Then, we get a strongly irreducible derivation π with the end sequent $\forall x F(x) \rightarrow$ by Theorem 1. Assume that π includes at least one boundary inference. Note that the end-place of π contains no free number variables. So, no *CJ*-inferences belong to the boundary of π. Thus each inference which belongs to the boundary of π must be of the form:

$$\frac{F(t), \Gamma \rightarrow \Theta}{\forall x F(x), \Gamma \rightarrow \Theta},$$

where Γ consists of $\forall x F(x)$ or formulas without logical symbols and Θ consists of formulas without logical symbols. Since π contains no redundant variables, t is a closed term. Let n be the value of t, then $\rightarrow n = t$ is derivable in *EA*. Therefore $\rightarrow F(t)$ is derivable in *EA*. But it contradicts the strong irreducibility of π. So, π includes no boundary inferences. Thus we can transform π into a derivation π' whose end sequent is a part of the end sequent of π and which includes no free variables, no weakenings, no essential cuts, no *CJ*-inferences and no logical inferences. Since any formula in π' does not include logical symbols, the end sequent of π' is $\rightarrow$. But, it is clear that there is not such a derivation. ■

4 Preliminaries

In this section , we shall define some necessary notions and state some propositions which can be proved easily.

4.1 Let Γ be a sequence $A_1, \cdots, A_n$ of formulas. Let $< i_1, \cdots, i_k >$ be a sequence of natural numbers such that $1 \leq i_1 < \cdots < i_k \leq n$. Then, the sequence $A_{i_1}, \cdots, A_{i_k}$ is called a *part* of Γ. Γ^* is used to denote a part of Γ. Let $\Gamma \rightarrow \Theta$ be a sequent. Then $\Gamma^* \rightarrow \Theta^*$ is called a *part* of $\Gamma \rightarrow \Theta$.

4.2 Let S be a sequent and S^* a part of S. And let π be a derivation of S and A a formula in π.

4.21 We say that A belongs to (S^*)-*ancestors* if a descendant of A is in S^* or an auxiliary formula of a *CJ*-inference in π.

4.22 A is said to be (S^*)-*explicit* if a descendant of A is in S, but not in S^*.

4.23 A is said to be *implicit* if a descendant of A is a cut formula of a cut in π.

4.3 Inductive definition of the *degree* $d(A)$ of a formula A:

4.31 $d(A) := 0$, if A contains no logical symbols.

4.32 $d(\neg A_1) := d(A_1) + 1$

4.33 $d(A_1 \wedge A_2) := d(A_1 \vee A_2) := max\{d(A_1), d(A_2)\} + 1$

4.34 $d(\forall x F(x)) := d(\exists x F(x)) := d(F(0)) + 1$

4.35 $d(\lambda x F(x)(t)) := d(F(0)) + 1$

4.36 $d(\forall X F(X)) := d(\exists X F(X)) := max\{\omega, d(F(U)) + 1\}$

Proposition 4.1. *The following hold for degrees of formulas:*

(1) $d(A) < \omega \cdot 2$.

(2) A *is elementary* $\Leftrightarrow d(A) < \omega$.

(3) $d(F(t)) = d(F(0))$.

(4) If $F(U)$ *contains at least one second order quantifier and* P *is an elementary predicator, then* $d(F(P)) = d(F(U))$.

(5) $d(A) < d(\neg A)$.

(6) $d(A_i) < d(A_1 \wedge A_2) = d(A_1 \vee A_2) (i = 1, 2)$.

(7) $d(F(t)) < d(\forall x F(x)) = d(\exists x F(x)) = d(\lambda x F(x)(t))$.

(8) If P is an elementary predicator, then $d(F(P)) < d(\forall X F(X))$.

4.4 Let I be an inference. Then the *degree* $d(I)$ of I is defined as follows:

$$d(I) := \begin{cases} \max\{d(A) : A \text{ is an auxliary formula of } I\}, & \text{if } I \text{ is a logical inference,} \\ \text{the degree of a cut formula of } I, & \text{if } I \text{ is a cut,} \\ \text{the degree of the induction formula of } I, & I \text{ is a } CJ\text{-inference,} \\ 0, & \text{otherwise.} \end{cases}$$

4.5 We define two functions on ordinals (cf. [8]):

4.51 Let ρ and σ be ordinals such that $\sigma \leq \rho$. Then the ordinal τ such that $\sigma + \tau = \rho$ is uniquely defined and is denoted by $-\sigma + \rho$.

4.52 Let ρ and α be ordinals and $\rho < \omega \cdot 2$. Then we define the ordinal function $\omega_\rho(\alpha)$ inductively as follows:

4.52.1 $\omega_0(\alpha) := \alpha$

4.52.2 $\omega_{n+1}(\alpha) := \omega^{\omega_n(\alpha)} \quad (n < \omega)$

4.52.3 $\omega_{\omega+n}(\alpha) := \varepsilon_{\omega_n(\alpha)} \quad (n < \omega)$

Proposition 4.2. *The following properties hold:*

(1) $\alpha < \beta \Rightarrow \omega_\rho(\alpha) < \omega_\rho(\beta)$

(2) $0 < \rho \ \& \ \beta, \gamma < \omega_\rho(\alpha) \Rightarrow \beta + 1, max\{\beta, \gamma\} + 1, \beta \sharp \gamma < \omega_\rho(\alpha)$

(3) $\omega \leq \rho \ \& \ \beta < \omega_\rho(\alpha) \Rightarrow \beta \cdot \omega, \omega^\beta < \omega_\rho(\alpha)$

(4) $1 \leq \beta \ \& \ n < \omega \Rightarrow \beta \cdot n < \beta \cdot \omega$

(5) $\sigma + \tau < \omega \cdot 2 \Rightarrow \omega_\sigma(\omega_\tau(\alpha)) = \omega_{\sigma+\tau}(\alpha)$
i.e. $\sigma \leq \rho (< \omega \cdot 2) \Rightarrow \omega_\sigma(\omega_{-\sigma+\rho}(\alpha)) = \omega_\rho(\alpha)$

4.6 Let π be a derivation and S a sequent in π. For any ordinal ρ such that $\rho < \omega \cdot 2$, the *height* $h_\rho(S;\pi)$ based on ρ of S in π is the maximum of

- ρ, or
- the degree of a cut or of a CJ-inference whose lower sequent stands below the sequent S, or
- the degree of a logical inference whose principal formula is not implicit and whose lower sequent stands below the sequent S.

Proposition 4.3.

$$\sigma \leq \rho \Rightarrow h_\sigma(S;\pi) \leq h_\rho(S;\pi).$$

4.7 Let π be a derivation with the end sequent $\check{S}$ and $\check{S}^*$ a part of $\check{S}$. And let ρ be an ordinal such that $\rho < \omega \cdot 2$. To each sequent S in π and each inference I in π, we assign ordinals $O_\rho(S;\pi;\check{S}^*)$, $O_\rho(I;\pi;\check{S}^*)$, $O_\rho(\pi;\check{S}^*)$, respectively, as follows:

4.71 Let S be an initial sequent.

4.71.1 If S is a logical initial sequent $U(t) \rightarrow U(t)$ or an equality initial sequent $s = t, U(s) \rightarrow U(t)$ and U is used as the eigenvariable of an inference in π and the auxiliary formula of the inference has at least one occurrence of U,

4.71.11 if at least one formula with U in the initial sequent is ($\check{S}^*$)-explicit,

$$O_\rho(S;\pi;\check{S}^*) := \omega^{\varepsilon_0+1};$$

4.71.12 if both formulas with U in the initial sequent are not ($\check{S}^*$)-explicit and at least one of them belongs to ($\check{S}^*$)-ancestors,

$$O_\rho(S;\pi;\check{S}^*) := \omega^2;$$

4.71.13 if both formulas with U in the initial sequent are implicit,

$$O_\rho(S;\pi;\check{S}^*) := \omega.$$

4.71.2 Otherwise,

$$O_\rho(S;\pi;\check{S}^*) := 1.$$

4.72 Let S_i $(1 \le i \le n)$ be the upper sequents of I. Assume that $O_\rho(S_i;\pi;\check{S}^*)$ is defined for each i $(1 \le i \le n)$.

4.72.1 If I is a weak inference,

$$O_\rho(I;\pi;\check{S}^*) := O_\rho(S_1;\pi;\check{S}^*).$$

4.72.2 If I is a logical inference,

4.72.21 if the principal formula of I is $(\check{S}^*)$-explicit and $h_\rho(S_i;\pi;\check{S}^*) \ge \omega$

$$O_\rho(I;\pi;\check{S}^*) := \begin{cases} O_\rho(S_1;\pi;\check{S}^*)\sharp\varepsilon_0, & \text{if I has one upper sequent,} \\ \max\{O_\rho(S_1;\pi;\check{S}^*), O_\rho(S_2;\pi;\check{S}^*)\}\sharp\varepsilon_0, & \text{if I has two upper sequents;} \end{cases}$$

4.72.22 if the principal formula of I is $(\check{S}^*)$-explicit and $h_\rho(S_i;\pi;\check{S}^*) < \omega$

$$O_\rho(I;\pi;\check{S}^*) := \begin{cases} O_\rho(S_1;\pi;\check{S}^*)\sharp\varepsilon_{\varepsilon_0}, & \text{if I has one upper sequent,} \\ \max\{O_\rho(S_1;\pi;\check{S}^*), O_\rho(S_2;\pi;\check{S}^*)\}\sharp\varepsilon_{\varepsilon_0}, & \text{if I has two upper sequents;} \end{cases}$$

4.72.23 if the principal formula of I belongs to $(\check{S}^*)$-ancestors,

$$O_\rho(I;\pi;\check{S}^*) := \begin{cases} O_\rho(S_1;\pi;\check{S}^*)\sharp(\omega\cdot 2), & \text{if I has one upper sequent,} \\ \max\{O_\rho(S_1;\pi;\check{S}^*), O_\rho(S_2;\pi;\check{S}^*)\}\sharp(\omega\cdot 2), & \text{if I has two upper sequents;} \end{cases}$$

4.72.24 the principal formula is implicit,

$$O_\rho(I;\pi;\check{S}^*) := \begin{cases} O_\rho(S_1;\pi;\check{S}^*)+1, & \text{if I has one upper sequent,} \\ \max\{O_\rho(S_1;\pi;\check{S}^*), O_\rho(S_2;\pi;\check{S}^*)\}+1, & \text{if I has two upper sequents.} \end{cases}$$

4.72.3 If I is a cut,

$$O_\rho(I;\pi;\check{S}^*) := O_\rho(S_1;\pi;\check{S}^*)\sharp O_\rho(S_2;\pi;\check{S}^*).$$

4.72.4 If I is a CJ-inference,

$$O_\rho(I;\pi;\check{S}^*) := O_\rho(S_1;\pi;\check{S}^*)\sharp(O_\rho(S_2;\pi;\check{S}^*)\cdot\omega)\sharp O_\rho(S_3;\pi;\check{S}^*)\sharp(\omega\cdot 2).$$

4.73 Let S be the lower sequent of I. And let σ be the height based on ρ of an upper sequent of I and τ the height based on ρ of S. Then,

$$O_\rho(S;\pi;\check{S}^*) := \omega_{-\tau+\sigma}(O_\rho(I;\pi;\check{S}^*)).$$

4.74 The ordinal of the derivation π:

$$O_\rho(\pi;\check{S}^*) := O_\rho(\check{S};\pi;\check{S}^*).$$

Proposition 4.4. *Let π be a derivation with the end sequent $\check{S}$ and $\check{S}^*$ a part of $\check{S}$.*

(1) Let S be a sequent in π and $\sigma = h_\rho(S;\pi)$, then

$$O_\rho(S;\pi;\check{S}^*) = O_\sigma(S;\pi;\check{S}^*).$$

(2) Let σ and ρ be ordinals such that $\sigma \le \rho(<\omega\cdot 2)$. And let S be a sequent in π , then

$$O_\sigma(S;\pi;\check{S}^*) \le \omega_{-h_\sigma(S;\pi)+h_\rho(S;\pi)}(O_\rho(S;\pi;\check{S}^*)).$$

(3) Let S be a sequent in π, then

(3.1) $h_\rho(S;\pi) \ge \omega \Rightarrow O_\rho(S;\pi;\check{S}^*) < \varepsilon_1$,

(3.2) $h_\rho(S;\pi) < \omega \Rightarrow O_\rho(S;\pi;\check{S}^*) < \varepsilon_{\varepsilon_1}$.

Proposition 4.5. *(1) The cut elimination theorem (with the exception of inessential cuts) for EA^- holds.*

(2) Let s and t be closed terms whose values are equal and sequent S be $F(s) \to F(t)$. Let π_0 be a standard derivation of S without essential cuts and π a derivation which contains π_0 as a subderivation. And let $F(s)$ and $F(t)$ in S are implicit in π. If $h_\rho(S;\pi) \ge d(F(s))$, then

$$O_\rho(S;\pi;\check{S}^*) < \omega\cdot 2.$$

(3) Let sequent S be $\rightarrow A$ *(or* $A \rightarrow$ *) and* π_0 *a derivation of S without essential cuts in* EA^- *and* π *a derivation which contains* π_0 *as a subderivation. And let the formula A in S is implicit. If* $h_\rho(S;\pi) \geq d(A)$, *then*

$$O_\rho(S;\pi;\check{S}^*) < \omega \cdot 2.$$

(4) Let sequent S be $\rightarrow A$ *(or* $A \rightarrow$ *) and* π_0 *a derivation of S in EA and* π *a derivation which contains* π_0 *as a subderivation. And let the formula A in S is implicit. If* $h_\rho(S;\pi) \geq d(A)$, *then*

(4.1) $h_\rho(S;\pi) \geq \omega \Rightarrow O_\rho(S;\pi;\check{S}^*) < \varepsilon_0$,

(4.2) $h_\rho(S;\pi) < \omega \Rightarrow O_\rho(S;\pi;\check{S}^*) < \varepsilon_{\varepsilon_0}$.

(5) Let P be an elementary predicator and S a sequent $P(t) \rightarrow P(t)$ *[* $s = t, P(s) \rightarrow P(t)$ *] where t is a term [s and t are terms]. Let* π_0 *be a standard derivation of S without essential cuts and* π *a derivation which contains* π_0 *as a subderivation. And let every free predicate variable in S is not used as the eigenvariable of any inference in* π. *If* $h_\rho(S;\pi) \geq \omega$, *then*

(5.1) if at least one of formulas with P in S is $(\check{S}^*)$*-explicit,*

$$O_\rho(S;\pi;\check{S}^*) < \omega^{\varepsilon_0+1};$$

(5.2) if both formulas with P in S are not $(\check{S}^*)$*-explicit and at least one of these belongs to* $(\check{S}^*)$*-ancestors,*

$$O_\rho(S;\pi;\check{S}^*) < \omega^2;$$

(5.3) if both formulas with P in S are implicit,

$$O_\rho(S;\pi;\check{S}^*) < \omega.$$

Proposition 4.6. *Let* π *be a derivation with the end sequent* $\Gamma_0 \rightarrow \Theta_0$ *and* $\Gamma \rightarrow \Theta$ *be a sequent belonging to the end-place of* π. *And let* π_1 *be the subderivation of* π *whose end sequent is* $\Gamma \rightarrow \Theta$ *and* π_2 *be a derivation with the end sequent* $\Gamma, \Delta \rightarrow \Lambda, \Theta$. *Let* Γ_0^*, Θ_0^*, Δ^* *and* Λ^* *be a part of* Γ_0, Θ_0, Δ *and* Λ, *respectively. Now we construct a derivation* π' *of* $\Gamma_0, \Delta \rightarrow \Lambda, \Theta_0$ *which is obtained from* π *by replacing* π_1 *by* π_2 *and adding formulas in* Δ, Λ *and some exchanges.*

$$\pi \begin{cases} \pi_1 \\ \vdots \\ \Gamma \rightarrow \Theta \\ \vdots \\ \Gamma_0 \rightarrow \Theta_0 \end{cases} \qquad \pi' \begin{cases} \pi_2 \\ \vdots \\ \Gamma, \Delta \rightarrow \Lambda, \Theta \\ \vdots \\ \Gamma_0, \Delta \rightarrow \Lambda, \Theta_0 \end{cases}$$

Suppose that $O_0(\Gamma, \Delta \rightarrow \Lambda, \Theta; \pi'; \Gamma_0^*, \Delta^* \rightarrow \Lambda^*, \Theta_0^*) < O_0(\Gamma \rightarrow \Theta; \pi; \Gamma_0^* \rightarrow \Theta_0^*)$, *then*

$$O_0(\pi'; \Gamma_0^*, \Delta^* \rightarrow \Lambda^*, \Theta_0^*) < O_0(\pi; \Gamma_0^* \rightarrow \Theta_0^*).$$

5 Proof of Theorem 1

5.1 Let π be a derivation with the end sequent $\check{S}$ in *EA* and $\check{S}^*$ a part of $\check{S}$. Then π is said to be $(\check{S}^*)$-*strongly irreducible* if it satisfies the following conditions:

5.11 It contains no cuts with the exception of inessential ones.

5.12 It contains no redundant variables.

5.13 It contains no inessential equality initial sequents.

5.14 It contains no *CJ*-inferences with the exception of constant normal ones.

5.15 Every logical inference, whose principal formula belongs to $(\check{S}^*)$-ancestors, in it is irreducible with respect to EA^-.

5.16 Every logical inference, whose principal formula is $(\check{S}^*)$-explicit, in it is irreducible with respect to *EA*.

Main Lemma. *Let* π *be a derivation with the end sequent* $\check{S}$ *and* $\check{S}^*$ *a part of S. We can transform* π *into an* $(\check{S}^*)$*-strongly irreducible derivation* π' *whose end sequent is* $\check{S}$.

Let S^* be the empty sequent $\rightarrow$, then Main Lemma implies Theorem 1.

We shall prove Main Lemma by induction on $O_0(\pi; \check{S}^*)$ ($< \varepsilon_{\varepsilon_1}$). Let $\check{S}$ be of the form $\Gamma_0 \rightarrow \Theta_0$ and $\check{S}^*$ of the form $\Gamma_0^* \rightarrow \Theta_0^*$.

5.2 As usual, we transform π into a derivation π' which satisfies the following conditions:

5.21 The end sequent of π' is $\check{S}$.

5.22 π' contains no redundant variables.

5.23 If I is a weakening in the end-place of π' , then every inference below I is an exchange or a weakening.

5.24 All eigenvariables in π' are distinct from one another. And if a free variable occurs as the eigenvariable in the upper sequent S of an inference in π', then it occurs only in sequents above S.

5.25 $O_0(\pi'; \check{S}^*) \leq O_0(\pi; \check{S}^*)$.

Therefore we may assume that π satisfies these conditions.

5.3 The case where π includes no boundary inferences.

Inferences in π are weak inferences and cuts. These cuts are only inessential by the assumption 5.2. If π have an inessential equality initial sequent, we can replace it by a logical or mathematical initial sequent with some weakenings. Thus we can transform π into a required derivation.

5.4 The case where π includes a boundary inference.

5.41 The case where π includes at least one CJ-inference as a boundary inference.
Assume that π is of the form:

$$\begin{array}{c}
\dfrac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma \xrightarrow{\alpha_1} \Theta, F(0)\end{array} \qquad \begin{array}{c}\pi_2(a) \\ \vdots \\ F(a), \Gamma \xrightarrow{\alpha_2} \Theta, F(a')\end{array} \qquad \begin{array}{c}\pi_3 \\ \vdots \\ F(t), \Gamma \xrightarrow{\alpha_3} \Theta\end{array}}{\Gamma \xrightarrow{\alpha} \Theta} I \\
\vdots \\
\Gamma_0 \rightarrow \Theta_0
\end{array}$$

where I belongs to the boundary of π and π_1 is the subderivation of the first upper sequent $S_1 : \Gamma \rightarrow \Theta, F(0)$; $\pi_2(a)$ is the subderivation of the second upper sequent $S_2 : F(a), \Gamma \rightarrow \Theta, F(a')$; π_3 is the subderivation of the third upper sequent $S_3 : F(t), \Gamma \rightarrow \Theta$ and the lower sequent of I is $S : \Gamma \rightarrow \Theta$. Assume that $h_0(S_1; \pi) = \rho, h_0(S; \pi) = \sigma$ and $\alpha_j = O_0(S_j; \pi; \check{S}^*)$ $(j = 1, 2, 3)$. Then

$$\alpha := O_0(S; \pi; \check{S}^*) = \omega_{-\sigma+\rho}(\alpha_1 \sharp (\alpha_2 \cdot \omega) \sharp \alpha_3 \sharp (\omega \cdot 2)).$$

5.41.1 The case where I is not constant normal.
We may assume that the induction formula $F(a)$ of I has at least one occurrence of a, since we can treat the other case similarly. Then the induction term t is closed. So, there is a numeral n such that $t = n$ is derivable in EA, and there exists a standard derivation π_0 , which dose not include essential cuts and CJ-inferences , of $F(n) \rightarrow F(t)$.
We shall reduce π into the following derivation π':

$$\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma \xrightarrow{\alpha_1'} \Theta, F(0)\end{array} \qquad \begin{array}{c}\pi_2(0) \\ \vdots \\ F(0), \Gamma \xrightarrow{\alpha_2'} \Theta, F(1)\end{array}}{\Gamma, \Gamma \rightarrow \Theta, \Theta, F(1)} \text{cut}}{\Gamma \rightarrow \Theta, F(1)} \\ \vdots \\ \Gamma \rightarrow \Theta, F(n)\end{array} \qquad \begin{array}{c}\pi_0 \\ \vdots \\ F(n) \xrightarrow{\alpha_0} F(t)\end{array}}{\Gamma \rightarrow \Theta, F(t)} \text{cut} \qquad \begin{array}{c}\pi_3 \\ \vdots \\ F(t), \Gamma \xrightarrow{\alpha_3'} \Theta\end{array}}{\dfrac{\Gamma, \Gamma \rightarrow \Theta, \Theta}{\Gamma \xrightarrow{\alpha'} \Theta}} \text{cut} \\
\vdots \\
\Gamma_0 \rightarrow \Theta_0
\end{array}$$

where $\pi_2(k)$ is obtained from $\pi_2(a)$ by substituting the numeral k for all occurrences of a $(k = 0, 1, \cdots, n-1)$. Let $\alpha'_j = O_0(S_j; \pi'; \check{S}^*)\,(j = 1, 3)$ and $\alpha_0 = O_0(F(n) \to F(t); \pi'; \check{S}^*)$. $h_0(S_1; \pi') = h_0(S_3; \pi') = \rho$. For all k , the ordinals $O_\rho(F(k), \Gamma \to \Theta, F(k'); \pi'; \check{S}^*)$ are the same value designated as α'_2.
For example, the formula $F(0)$ and its ancestors in the subderivation π_1 belong to $(\check{S}^*)$-ancestors in π and those are implicit in π'. So, the ordinals of some initial sequents and inferences in π_1 as a subderivation of π' may be smaller than ones in π. Therefore

$$\alpha'_1 = O_0(S_1; \pi'; \check{S}^*) = O_\rho(S_1; \pi'; \check{S}^*) \leq O_\rho(S_1; \pi; \check{S}^*) = O_0(S_1; \pi; \check{S}^*) = \alpha_1.$$

Similarly, We can prove $\alpha'_2 \leq \alpha_2$ and $\alpha'_3 \leq \alpha_3$. And $\alpha_0 < \omega \cdot 2$ by Proposition 4.5 (2). Thus,

$$\begin{aligned} \alpha' := O_0(S; \pi'; \check{S}^*) &= \omega_{-\sigma+\rho}(\alpha'_1 \sharp (\alpha'_2 \cdot n) \sharp \alpha'_3 \sharp \alpha_0) \\ &\leq \omega_{-\sigma+\rho}(\alpha_1 \sharp (\alpha_2 \cdot n) \sharp \alpha_3 \sharp \alpha_0) \\ &< \omega_{-\sigma+\rho}(\alpha_1 \sharp (\alpha_2 \cdot \omega) \sharp \alpha_3 \sharp (\omega \cdot 2)) \\ &= O_0(S; \pi; \check{S}^*) = \alpha. \end{aligned}$$

So, $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$ by Proposition 4.6. Therefore we can transform π' into an $(\check{S}^*)$-strongly irreducible derivation with the end sequent $\check{S}$ by induction hypothesis.

5.41.2 The case where I is constant normal.
We shall construct the following derivations $\hat{\pi}_1, \hat{\pi}_2$ and $\hat{\pi}_3$ from π:

$$\hat{\pi}_1 \left\{ \begin{array}{c} \pi_1 \\ \vdots \\ \Gamma \to \Theta, F(0) \\ \hline\hline \Gamma \to F(0), \Theta \\ \vdots \\ \Gamma_0 \to F(0), \Theta_0 \\ \hline\hline \Gamma_0 \to \Theta_0, F(0) \end{array} \right. \qquad \hat{\pi}_2 \left\{ \begin{array}{c} \pi_2(a) \\ \vdots \\ F(a), \Gamma \to \Theta, F(a') \\ \hline\hline \Gamma, F(a) \to F(a'), \Theta \\ \vdots \\ \Gamma_0, F(a) \to F(a'), \Theta_0 \\ \hline\hline F(a), \Gamma_0 \to \Theta_0, F(a') \end{array} \right. \qquad \hat{\pi}_3 \left\{ \begin{array}{c} \pi_3 \\ \vdots \\ F(t), \Gamma \to \Theta \\ \hline\hline \Gamma, F(t) \to \Theta \\ \vdots \\ \Gamma_0, F(t) \to \Theta_0 \\ \hline\hline F(t), \Gamma_0 \to \Theta_0 \end{array} \right.$$

We shall prove $O_0(\hat{\pi}_1; \Gamma_0^* \to \Theta_0^*, F(0)) < O_0(\pi; \check{S}^*)$. $h_0(S_1, \hat{\pi}_1) = \sigma$. For each formula in $\hat{\pi}_1$, it is $(\Gamma_0^* \to \Theta_0^*, F(0))$-explicit or implicit or belongs to $(\Gamma_0^* \to \Theta_0^*, F(0))$-ancestors if and only if the formula corresponding to it in π is $(\check{S}^*)$-explicit or implicit or belongs to $(\check{S}^*)$-ancestors, respectively. So, by Proposition 4.4 (2),

$$\begin{aligned} O_0(S_1; \hat{\pi}_1; \Gamma_0^* \to \Theta_0^*, F(0)) &= O_\sigma(S_1; \hat{\pi}_1; \Gamma_0^* \to \Theta_0^*, F(0)) \\ &= O_\sigma(S_1; \pi_1; \Gamma_0^* \to \Theta_0^*, F(0)) \\ &\leq \omega_{-\sigma+\rho}(O_\rho(S_1; \pi_1; \Gamma_0^* \to \Theta_0^*, F(0))) \\ &= \omega_{-\sigma+\rho}(O_\rho(S_1; \pi_1; \check{S}^*)) = \omega_{-\sigma+\rho}(O_0(S_1; \pi_1; \check{S}^*)). \end{aligned}$$

Hence,

$$\begin{aligned} & O_0\,(\Gamma \to F(0), \Theta; \hat{\pi}_1; \Gamma_0^* \to \Theta_0^*, F(0)) \\ & = O_0(S_1; \hat{\pi}_1; \Gamma_0^* \to \Theta_0^*, F(0)) \\ & \leq \omega_{-\sigma+\rho}(O_0(S_1; \pi_1; \check{S}^*)) \\ & < \omega_{-\sigma+\rho}(O_0(S_1; \pi_1; \check{S}^*) \sharp (O_0(S_2; \pi_1; \check{S}^*) \cdot \omega) \sharp O_0(S_3; \pi_1; \check{S}^*) \sharp (\omega \cdot 2)) \\ & = O_0(S; \pi; \check{S}^*). \end{aligned}$$

Therefore, $O_0(\hat{\pi}_1; \Gamma_0^* \to \Theta_0^*, F(0)) < O_0(\pi; \check{S}^*)$ by Proposition 4.6. Similarly, we can prove $O_0(\hat{\pi}_2; F(a), \Gamma_0^* \to \Theta_0^*, F(a')) < O_0(\pi; \check{S}^*)$ and $O_0(\hat{\pi}_3; F(t), \Gamma_0^* \to \Theta_0^*) < O_0(\pi; \check{S}^*)$.
Thus, by induction hypothesis, we can transform $\hat{\pi}_1$ into a $(\Gamma_0^* \to \Theta_0^*, F(0))$-strongly irreducible derivation π'_1 whose end sequent is $\Gamma_0 \to \Theta_0, F(0)$ and $\hat{\pi}_2$ into a $(F(a), \Gamma_0^* \to \Theta_0^*, F(a'))$-strongly irreducible derivation π'_2 whose end sequent is $F(a), \Gamma_0 \to \Theta_0, F(a')$ and $\hat{\pi}_3$ into a $(F(t), \Gamma_0^* \to \Theta_0^*)$-strongly irreducible derivation π'_3 whose end sequent is $F(t), \Gamma_0 \to \Theta_0$. We shall define π' as follows:

$$\dfrac{\begin{array}{c} \pi'_1 \\ \vdots \\ \Gamma_0 \to \Theta_0, F(0) \end{array} \quad \begin{array}{c} \pi'_2 \\ \vdots \\ F(a), \Gamma_0 \to \Theta_0, F(a') \end{array} \quad \begin{array}{c} \pi'_3 \\ \vdots \\ F(t), \Gamma_0 \to \Theta_0 \end{array}}{\Gamma_0 \to \Theta_0}\ J$$

Note that π includes no redundant variables, I is constant normal and belongs to the boundary of π, and a does not occur in $F(t)$ and $\Gamma_0 \to \Theta_0$ by our assumption 5.24. So, the free number variables included in t occur in $\Gamma_0 \to \Theta_0$ and J is a constant normal CJ-inference. π' is an $(\check{S}^*)$-strongly irreducible derivation whose end sequent is $\check{S}$.

5.42 The case where π includes at least one boundary logical inference whose principal formula belongs to $(\check{S}^*)$-ancestors.

Let I be such a inference. We treat only the case that I is a $\forall^2$-IS. All the other cases can be treated similarly. Assume that π is of the form:

$$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\Gamma \xrightarrow{\alpha_1} \Theta, F(U)}{\Gamma \xrightarrow{\alpha} \Theta, \forall X F(X)}\ I \\ \vdots \\ \Gamma_0 \to \Theta_0 \end{array}$$

where I belongs to the boundary of π and π_1 is the subderivation of the upper sequent $S_1 : \Gamma \to \Theta, F(U)$ and the lower sequent of I is $S : \Gamma \to \Theta, \forall X F(X)$. Assume that $h_0(S_1;\pi) = \rho, h_0(S;\pi) = \sigma$ and $\alpha_1 = O_0(S_1;\pi;\check{S}^*)$. Then $\rho \geq d(F(U))$ and

$$\alpha := O_0(S;\pi;\check{S}^*) = \omega_{-\sigma+\rho}(\alpha_1 \sharp (\omega \cdot 2)).$$

5.42.1 The case where I is reducible with respect to EA^-.

By our assumption, the sequent $F(U) \to$ is derivable in EA^-. Let $\hat{\pi}$ be a derivation of $F(U) \to$ without essential cuts. Then we shall reduce π into the following derivation π':

$$\begin{array}{c} \begin{array}{cc} \pi_1 & \hat{\pi} \\ \vdots & \vdots \end{array} \\ \dfrac{\dfrac{\Gamma \xrightarrow{\alpha_1'} \Theta, F(U) \qquad F(U) \xrightarrow{\beta}}{\Gamma \to \Theta}\ \text{cut}}{\Gamma \xrightarrow{\alpha'} \Theta, \forall X F(X)}\ \text{weakening} \\ \vdots \\ \Gamma_0 \to \Theta_0 \end{array}$$

$h_0(S_1;\pi') = \rho$, $h_0(S;\pi') = \sigma$. Let $\alpha_1' = O_0(S_1;\pi';\check{S}^*)$ and $\beta = O_0(F(U) \to;\pi';\check{S}^*)$. The free predicate variable U is used as the eigenvariable of the inference I in π , but it is not used as the eigenvariable of any inferences in π'. And the formula $F(U)$ and its ancestors in π belong to $(\check{S}^*)$-ancestors and those in π' are implicit. So, the ordinals of some initial sequents and inferences in π' may be smaller than ones in π. Thus, $\alpha_1' \leq \alpha_1$. And $\beta < \omega \cdot 2$ by Proposition 4.5 (3). Therefore,

$$\alpha' := O_0(S;\pi';\check{S}^*) = \omega_{-\sigma+\rho}(\alpha_1' \sharp \beta) \leq \omega_{-\sigma+\rho}(\alpha_1 \sharp \beta) < \omega_{-\sigma+\rho}(\alpha_1 \sharp (\omega \cdot 2)) = \alpha.$$

So, $O_0(\pi';\check{S}^*) < O_0(\pi;\check{S}^*)$ by Proposition 4.6. We can transform π' into an $(\check{S}^*)$-strongly irreducible derivation whose end sequent is $\check{S}$ by induction hypothesis.

5.42.2 The case where I is irreducible with respect to EA^-.

We shall construct the following derivation $\hat{\pi}$ from π:

$$\begin{array}{c} \pi_1 \\ \vdots \\ \dfrac{\dfrac{\Gamma \xrightarrow{\alpha_1'} \Theta, F(U)}{\Gamma \to F(U), \Theta}}{\Gamma \xrightarrow{\alpha'} F(U), \Theta, \forall X F(X)}\ \text{weakening} \\ \vdots \\ \Gamma_0 \to F(U), \Theta_0 \end{array}$$

$h_0(S_1;\hat{\pi}) = h_0(\Gamma \to F(U), \Theta, \forall X F(X);\hat{\pi}) = \sigma$. Let $\hat{S}$ be the end sequent $\Gamma_0 \to F(U), \Theta_0$ of $\hat{\pi}$ and $\hat{S}^*$ be the part of $\hat{S}$ which is obtained from $\Gamma_0 \to F(U), \Theta_0$ by deleting $(\check{S}^*)$-explicit formulas in $\check{S}$. Especially, the formula $F(U)$ belongs to $(\hat{S}^*)$-ancestors in $\hat{\pi}$. Note the formula $F(U)$ and its ancestors in π belong to $(\check{S}^*)$-ancestors.

Let $\alpha_1' = O_0(S_1;\hat{\pi};\check{S}^*)$. The free predicate variable U is used as the eigenvariable of the inference I in π , but it is not used as the eigenvariable of any inferences in $\hat{\pi}$. So, the ordinals of some initial sequents and inferences in $\hat{\pi}$ may be smaller than ones in π. Thus, $\alpha_1' \leq \alpha_1$. Therefore,

$$\begin{aligned}\alpha' := O_0(\Gamma \to F(U), \Theta, \forall X F(X);\hat{\pi};\hat{S}^*) = \alpha_1' \\ \leq \omega_{-\sigma+\rho}(\alpha_1') \leq \omega_{-\sigma+\rho}(\alpha_1) \\ < \omega_{-\sigma+\rho}(\alpha_1 \sharp (\omega \cdot 2)) = \alpha.\end{aligned}$$

So, $O_0(\hat{\pi};\hat{S}^*) < O_0(\pi;\check{S}^*)$ by Proposition 4.6. We can transform $\hat{\pi}$ into an $(\hat{S}^*)$-strongly irreducible derivation $\hat{\pi}'$ whose end sequent is $\Gamma_0 \to F(U), \Theta_0$ by induction hypothesis. We shall define π' whose end sequent is $\check{S}$ as follows:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\hat{\pi}' \\ \vdots \\ \Gamma_0 \to F(U), \Theta_0\end{array}}{\Gamma_0 \to \Theta_0, F(U)}}{\Gamma_0 \to \Theta_0, \forall X F(X)}\ J}{\Gamma_0 \to \Theta_0}$$

The free predicate variable U does not occur in $\Gamma_0 \to \Theta_0, \forall X F(X)$ by the assumption 5.24, because U is used as the eigenvariable of I in π. So, the inference J whose principal formula belongs to $(\check{S}^*)$-ancestors in π' satisfies the restrictions on predicate variables for $\forall^2$-IS . And the sequent $F(U) \to$ is not derivable in EA^- by our assumption. Thus π' is an $(\check{S}^*)$-strongly irreducible derivation whose end sequent is $\check{S}$.

5.43 The case where π includes at least one boundary logical inference whose principal formula is $(\check{S}^*)$-explicit.

Let I be such a inference. We treat only the case that I is a $\forall^2$-IS. All the other cases can be treated similarly. Assume that π is of the form:

$$\begin{array}{c}\pi_1 \\ \vdots \\ \dfrac{\Gamma \xrightarrow{\alpha_1} \Theta, F(U)}{\Gamma \xrightarrow{\alpha} \Theta, \forall X F(X)}\ I \\ \vdots \\ \Gamma_0 \to \Theta_0\end{array}$$

where I belongs to the boundary of π and π_1 is the subderivation of the upper sequent $S_1 : \Gamma \to \Theta, F(U)$ and the lower sequent of I is $S : \Gamma \to \Theta, \forall X F(X)$. Assume that $h_0(S_1;\pi) = \rho, h_0(S;\pi) = \sigma$ and $\alpha_1 = O_0(S_1;\pi;\check{S}^*)$. Then $\rho \geq d(F(U))$ and

$$\alpha = O_0(S;\pi;\check{S}^*) = \begin{cases}\omega_{-\sigma+\rho}(\alpha_1 \sharp \varepsilon_0), & \text{if } \rho \geq \omega, \\ \omega_{-\sigma+\rho}(\alpha_1 \sharp \varepsilon_{\varepsilon_0}), & \text{if } \rho < \omega.\end{cases}$$

5.43.1 The case where I is reducible with respect to EA.

By our assumption, the sequent $F(U) \to$ is derivable in EA. Let $\hat{\pi}$ be a derivation of $F(U) \to$.

Then we shall reduce π into the following derivation π':

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \xrightarrow{\alpha_1'} \Theta, F(U)\end{array} \qquad \begin{array}{c}\hat{\pi}\\ \vdots\\ F(U) \xrightarrow{\beta}\end{array}}{\Gamma \to \Theta}\ \text{cut}}{\begin{array}{c}\Gamma \xrightarrow{\alpha'} \Theta, \forall X F(X)\\ \vdots\\ \Gamma_0 \to \Theta_0\end{array}}\ \text{weakening}$$

$h_0(S_1;\pi') = \rho$, $h_0(S;\pi') = \sigma$. Let $\alpha_1' = O_0(S_1;\pi';\check{S}^*)$ and $\beta = O_0(F(U) \to;\pi';\check{S}^*)$. The free predicate variable U is used as the eigenvariable of the inference I in π , but it is not used as the eigenvariable of any inferences in π'. And the formula $F(U)$ and its ancestors in π are $(\check{S}^*)$-explicit and those in π' are implicit. So, the ordinals of some initial sequents and inferences in π' may be smaller than ones in π. Thus, $\alpha_1' \leq \alpha_1$. And if $\rho \geq \omega$ then $\beta < \varepsilon_0$ and if $\rho < \omega$ then $\beta < \varepsilon_{\varepsilon_0}$ by Proposition 4.5 (4). Therefore,

$$\alpha' := O_0(S;\pi';\check{S}^*) = \omega_{-\sigma+\rho}(\alpha_1' \sharp \beta) \leq \omega_{-\sigma+\rho}(\alpha_1 \sharp \beta) < \omega_{-\sigma+\rho}(\alpha_1 \sharp \varepsilon_\gamma) = \alpha,$$

where if $\rho \geq \omega$ then $\gamma = 0$ and if $\rho < \omega$ then $\gamma = \varepsilon_0$.
So, $O_0(\pi';\check{S}^*) < O_0(\pi;\check{S}^*)$ by Proposition 4.6. We can transform π' into an $(\check{S}^*)$-strongly irreducible derivation whose end sequent is $\check{S}$ by induction hypothesis.

5.43.2 The case where I is irreducible with respect to EA.
We shall construct the following derivation $\hat{\pi}$ from π:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma \xrightarrow{\alpha_1'} \Theta, F(U)\\ \hline\hline \Gamma \to F(U), \Theta\end{array}}{\begin{array}{c}\Gamma \xrightarrow{\alpha'} F(U), \Theta, \forall X F(X)\\ \vdots\\ \Gamma_0 \to F(U), \Theta_0\end{array}}\ \text{weakening}$$

$h_0(S_1;\hat{\pi}) = h_0(\Gamma \to F(U), \Theta, \forall X F(X);\hat{\pi}) = \sigma$. Let $\hat{S}$ be the end sequent $\Gamma_0 \to F(U), \Theta_0$ of $\hat{\pi}$ and $\hat{S}^*$ be the part of $\hat{S}$ which is obtained from $\Gamma_0 \to F(U), \Theta_0$ by deleting the formula $F(U)$ and $(\check{S}^*)$-explicit formulas in $\check{S}$. Especially, the formula $F(U)$ is $(\hat{S}^*)$-explicit in $\hat{\pi}$. Note the formula $F(U)$ and its ancestors in π are $(\check{S}^*)$-explicit.
Let $\alpha_1' = O_0(S_1;\hat{\pi};\check{S}^*)$. The free predicate variable U is used as the eigenvariable of the inference I in π , but it is not used as the eigenvariable of any inferences in $\hat{\pi}$. So, the ordinals of some initial sequents and inferences in $\hat{\pi}$ may be smaller than ones in π. Thus, $\alpha_1' \leq \alpha_1$. Therefore,

$$\begin{aligned}\alpha' := O_0(\Gamma \to F(U), \Theta, \forall X F(X);\hat{\pi};\hat{S}^*) &= \alpha_1'\\ &\leq \omega_{-\sigma+\rho}(\alpha_1') \leq \omega_{-\sigma+\rho}(\alpha_1)\\ &< \omega_{-\sigma+\rho}(\alpha_1 \sharp \varepsilon_\gamma) = \alpha,\end{aligned}$$

where if $\rho \geq \omega$ then $\gamma = 0$ and if $\rho < \omega$ then $\gamma = \varepsilon_0$. So, $O_0(\hat{\pi};\hat{S}^*) < O_0(\pi;\check{S}^*)$ by Proposition 4.6. We can transform $\hat{\pi}$ into an $(\hat{S}^*)$-strongly irreducible derivation $\hat{\pi}'$ whose end sequent is $\Gamma_0 \to F(U), \Theta_0$ by induction hypothesis. We shall define π' whose end sequent is $\check{S}$ as follows:

$$\dfrac{\begin{array}{c}\hat{\pi}'\\ \vdots\\ \Gamma_0 \to F(U), \Theta_0\\ \hline\hline \Gamma_0 \to \Theta_0, F(U)\end{array}}{\begin{array}{c}\Gamma_0 \to \Theta_0, \forall X F(X)\\ \hline\hline \Gamma_0 \to \Theta_0\end{array}}\ J$$

The free predicate variable U does not occur in $\Gamma_0 \to \Theta_0, \forall X F(X)$ by the assumption 5.24, because U is used as the eigenvariable of I in π. So, the inference J whose principal formula is $(\check{S}^*)$-explicit in π' satisfies the restrictions on predicate variables for $\forall^2$-IS . And the sequent $F(U) \to$ is not derivable in EA by our assumption. Thus π' is an $(\check{S}^*)$-strongly irreducible derivation whose end sequent is $\check{S}$.

5.44 The case where every boundary inference of π is a logical inference whose principal formula is implicit. We shall prove that there is a suitable cut (cf. [11], [3]).

Let $\mathfrak{M}$ be the whole of sequents which are below a boundary logical inference. Now we consider the following property $(*)$ for a sequent of $\mathfrak{M}$.

(*) It includes a descendant of the principal formula of a boundary logical inference.

The lower sequent of a boundary logical inference satisfies the property $(*)$ and the end sequent $\check{S}$ does not. Therefore there is a sequent in $\mathfrak{M}$ which dose not satisfy the property $(*)$ and the sequent(s) above it in $\mathfrak{M}$ satisfies the property $(*)$. We take one of the uppermost sequent in such ones and denote it by S. It is clear that S is the lower sequent of a cut. We denote it by I. Let S_1 and S_2 be the upper sequents of I. We may assume that S_1 belongs to $\mathfrak{M}$ and S_1 satisfy the property $(*)$. Then the cut formula which is included in S_1 must be a descendant of the principal formula of a boundary logical inference and include logical symbols, because the lower sequent S of I does not satisfy the property $(*)$. Suppose that S_2 does not belong to $\mathfrak{M}$, then all the uppermost sequents above S_2 are initial sequents in the end-place. There is no weakening above S_2 by our assumption 5.23. So, S_2 can not include formulas with logical symbols. It is contradictory to that S_2 includes the cut formula of I. Therefore S_2 belongs to $\mathfrak{M}$ and S_2 satisfies the property $(*)$. The cut formula which is included in S_2 is a descendant of the principal formula of a boundary logical inference. The cut I is a suitable cut.

We shall treat the case that the cut formula of I is of the form $\forall X F(X)$. All the other cases can be treated similarly.

Assume that π is of the form:

$$
\begin{array}{lccc}
 & U(t) \xrightarrow{\gamma} U(t) & & \pi_2 \\
\pi_1(U) & \vdots & & \vdots \\
 & \dfrac{\Gamma_1 \to \Theta_1, F(U)}{\Gamma_1 \to \Theta_1, \forall X F(X)}\, I_1 & & \dfrac{F(P), \Gamma_2 \to \Theta_2}{\forall X F(X), \Gamma_2 \to \Theta_2}\, I_2 \\
 & \vdots & & \vdots \\
\text{height } \rho & \Gamma \to \Theta, \forall X F(X) & & \forall X F(X), \Delta \to \Lambda \\
\hline
 & & \Gamma, \Delta \to \Theta, \Lambda \quad I & \\
\text{height } \rho & & \vdots & \\
\text{height } \sigma < \rho & & \overline{\Gamma_3 \to \Theta_3}\; I_3; \text{ordinal } \alpha & \\
 & & \vdots & \\
 & & \Gamma_0 \to \Theta_0 &
\end{array}
$$

where I is a suitable cut whose cut formula is $\forall X F(X)$ and I_1 is a boundary logical inference of π whose principal formula is an ancestor of the cut formula of the left upper sequent of I and I_2 is a boundary logical inference of π whose principal formula is an ancestor of the cut formula of the right upper sequent of I. Let $\pi_1(U)$ be the subderivation of π whose end sequent is the upper sequent $\Gamma_1 \to \Theta_1, F(U)$ of I_1 and π_2 be the subderivation of π whose end sequent is the upper sequent $F(P), \Gamma_2 \to \Theta_2$ of I_2. The free predicate variable U is used as the eigenvariable of I_1 and occurs only in $\pi_1(U)$ by our assumption 5.24. P is a elementary predicator. Let ρ be the height based on 0 of the upper sequents of I:

$$\rho = h_0(\Gamma \to \Theta, \forall X F(X); \pi) = h_0(\forall X F(X), \Delta \to \Lambda; \pi).$$

Let $\Gamma_3 \to \Theta_3$ be the uppermost sequent below I whose height based on 0 is less than ρ and I_3 be the inference whose lower sequent is $\Gamma_3 \to \Theta_3$. Note $\rho \geq d(I) = d(\forall X F(X)) \geq \omega$ and $h_0(\Gamma_0 \to \Theta_0; \pi) = 0$. So, there is such a sequent. Let $\sigma = h_0(\Gamma_3 \to \Theta_3; \pi)$ and $\alpha = O_0(I_3; \pi; \check{S}^*)$. The subderivation $\pi_1(U)$ may include an initial sequent $S_0(U)$ with U which is a logical initial sequent $U(t) \to U(t)$ or

an equality initial sequent $s = t, U(s) \to U(t)$. But if $F(U)$ has no occurrences of U, then there are no such initial sequents in $\pi_1(U)$ by our assumption 5.22. Let $\gamma = O_0(S_0(U); \pi; \check{S}^*)$.
We shall reduce π into the following derivation π':

$$
\begin{array}{c}
\pi_0(P) \\
\vdots \\
P(t) \xrightarrow{\gamma'} P(t) \\
\vdots \\
\Gamma_1 \to \Theta_1, F(P)
\end{array}
\Bigg\} \pi_1(P)
\qquad\qquad
\begin{array}{c}
\pi_2 \\
\vdots \\
F(P), \Gamma_2 \to \Theta_2
\end{array}
$$

$$
\dfrac{\Gamma_1 \to \Theta_1, F(P)}{\Gamma_1 \to F(P), \Theta_1, \forall X F(X)}
\qquad\qquad
\dfrac{F(P), \Gamma_2 \to \Theta_2}{\forall X F(X), \Gamma_2, F(P) \to \Theta_2}
$$

$$
\dfrac{\Gamma \to F(P), \Theta, \forall X F(X) \qquad \forall X F(X), \Delta \to \Lambda}{\Gamma, \Delta \to F(P), \Theta, \Lambda}
\qquad
\dfrac{\Gamma \to \Theta, \forall X F(X) \qquad \forall X F(X), \Delta, F(P) \to \Lambda}{\Gamma, \Delta, F(P) \to \Theta, \Lambda}
$$

$$
\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_3 \to F(P), \Theta_3}\, I_3^1; \alpha_1}{\Gamma_3 \to \Theta_3, F(P)}
\qquad
\dfrac{\dfrac{\vdots}{\Gamma_3, F(P) \to \Theta_3}\, I_3^2; \alpha_2}{F(P), \Gamma_3 \to \Theta_3}}
{\dfrac{\Gamma_3, \Gamma_3 \to \Theta_3, \Theta_3}{\Gamma_3 \to \Theta_3}\, \alpha'}\, J
$$

$$
\begin{array}{c}
\vdots \\
\Gamma_0 \to \Theta_0
\end{array}
$$

where $\pi_1(P)$ is obtained from $\pi_1(U)$ by replacing $U(t)$ by $P(t)$ and $\pi_0(P)$ is a standard derivation of the sequent $S_0(P)$ which is obtained from a initial sequent $S_0(U)$ by replacing $U(t)$ by $P(t)$. J is the new cut whose cut formula is $F(P)$ and all the ancestors of $F(P)$ are implicit in π'. I_3^1 and I_3^2 are the inferences corresponding to I_3.
We shall prove $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$. The predicator P may include a free predicate variable V and the derivation $\pi_0(P)$ may have a initial sequent with V. Note that π has no redundant variables by our assumption 5.22. So, the free variable V included in P which occurs in the upper sequent of the boundary inference of π must occur in the end sequent $\Gamma_0 \to \Theta_0$. Therefore, no free variables included in P is used as the eigenvariable in π'. Note $d(P(t)) < \omega \le d(\forall X F(X)) \le \rho \le h_0(S_0(P); \pi'; \check{S}^*)$. Let $\gamma' = O_0(S_0(P); \pi'; \check{S}^*)$, then $\gamma' \le \gamma$ by Proposition 4.5(5) and 4.71. And the heights based on 0 of the sequents in $\pi_1(U)$ and $\pi_1(P)$ corresponding mutually have the same values, because $h_0(\Gamma_1 \to \Theta_1, \forall X F(X); \pi) \ge \rho \ge \omega$ and for any formula $G(U)$ in $\pi_1(U)$, if $d(G(U)) < \omega$ then $d(G(P)) < \omega$ and if $d(G(U)) \ge \omega$ then $d(G(P)) = d(G(U))$ by Proposition 4.1. The replacement of U in $\pi_1(U)$ by P does not increase the height based on 0 of a sequent in $\pi_1(U)$.
Since the ordinal assignments for $\pi_1(U)$ and $\pi_1(P)$ go side by side,

$$O_0(\Gamma_1 \to \Theta_1, F(P); \pi'; \check{S}^*) \le O_0(\Gamma_1 \to \Theta_1, F(U); \pi; \check{S}^*).$$

Therefore,

$$
\begin{aligned}
O_0(\Gamma_1 \to F(P), \Theta_1, \forall X F(X); \pi'; \check{S}^*) &= O_0(\Gamma_1 \to \Theta_1, F(P); \pi'; \check{S}^*) \\
&\le O_0(\Gamma_1 \to \Theta_1, F(U); \pi; \check{S}^*) \\
&< O_0(\Gamma_1 \to \Theta_1, F(U); \pi; \check{S}^*) + 1 \\
&= O_0(\Gamma_1 \to \Theta_1, \forall X F(X); \pi; \check{S}^*).
\end{aligned}
$$

Hence,

$$\alpha_1 := O_0(I_3^1; \pi'; \check{S}^*) < O_0(I_3; \pi; \check{S}^*) = \alpha.$$

Similarly , $\alpha_2 := O_0(I_3^2; \pi'; \check{S}^*) < O_0(I_3; \pi; \check{S}^*) = \alpha$. Let

$$\tau = h_0(\Gamma_3 \to F(P), \Theta_3; \pi') = h_0(\Gamma_3, F(P) \to \Theta_3; \pi').$$

Now,

$$
\begin{aligned}
O_0(\Gamma_3 \to \Theta_3, F(P); \pi'; \check{S}^*) &= \omega_{-\tau+\rho}(\alpha_1) < \omega_{-\tau+\rho}(\alpha), \\
O_0(F(P), \Gamma_3 \to \Theta_3; \pi'; \check{S}^*) &= \omega_{-\tau+\rho}(\alpha_2) < \omega_{-\tau+\rho}(\alpha).
\end{aligned}
$$

By the definition, $\tau = \max\{\sigma, d(F(P))\}$. So, $\sigma \le \tau < \rho$ and $-\tau + \rho > 0$. Therefore,

$$O_0(\Gamma_3 \to \Theta_3, F(P); \pi'; \check{S}^*) \sharp O_0(F(P), \Gamma_3 \to \Theta_3; \pi'; \check{S}^*) < \omega_{-\tau+\rho}(\alpha).$$

Thus,

$$\begin{aligned} \alpha' &:= \omega_{-\sigma+\tau}\left(O_0(\Gamma_3 \to \Theta_3, F(P); \pi'; \check{S}^*) \sharp O_0(F(P), \Gamma_3 \to \Theta_3; \pi'; \check{S}^*)\right) \\ &< \omega_{-\sigma+\tau}\left(\omega_{-\tau+\rho}(\alpha)\right) = \omega_{-\sigma+\rho}(\alpha). \end{aligned}$$

We have

$$O_0(\Gamma_3 \to \Theta_3; \pi', \check{S}^*) = \alpha' < \omega_{-\sigma+\rho}(\alpha) = O_0(\Gamma_3 \to \Theta_3; \pi, \check{S}^*).$$

So, $O_0(\pi'; \check{S}^*) < O_0(\pi; \check{S}^*)$ by Proposition 4.6. We can transform π' into an $(\check{S}^*)$-strongly irreducible derivation whose end sequent is $\check{S}$ by induction hypothesis. ■

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初等解析の標準形定理

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概要 : 数についてペアノの公理系が成り立ち、述語変数に対する量化が存在する形式的体系は古典解析と呼ばれている。この体系では実数は有理数のある種の集合として述語子で定義され、実数についての全称および存在命題が形式化可能である。初等解析の形式的体系とは古典解析の部分体系であって、その中に述語変数に対する量化子が含まれていないような初等述語子にだけ内包公理を制限したものである。また、ある形式的体系の ω 無矛盾性はその体系の単なる無矛盾性よりも強い性質である。その体系自身の ω 無矛盾性を導ける初等解析の形式的体系についてのある標準形定理が $\varepsilon_{\varepsilon_1}$ までの超限帰納法を用いて証明されることを示す。