

# Concrete Arrangement of Subgroups in the Exceptional Lie Group $F_4^C$ Using the Yokota-style Method

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(Received: May 31st, 2021)

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**Abstract:** In this research, we use Jordan algebra, split Jordan algebra and their complexifications specifically for constructing subgroups of the exceptional Lie group  $F_4^C$  in the Yokota-style. The Yokota-style algebraic construction method characterizes non-compact groups and complex Lie groups as being constructed naturally. First, we define Cayley algebra to consider Lie groups of type  $G_2$ , and then we use Jordan algebra to extend them to Lie groups of type  $F_4$  and their involutions. Finally, we consider the arrangement of the subgroups of type  $F_4$  Lie groups using the involutions. In particular, we consider the arrangement of two non-compact groups of type  $F_4$  and their subgroups. We specifically use the involutions and construct subgroups as their invariant subgroups.

## 1. Introduction

E. Cartan mentioned without proof that octonions and the  $G_2$  type Lie group are related, and N. Jacobson constructed a non-compact Lie algebra of type  $G_2$  using split octonions. Later,  $F_4$  and  $E_6$  were given by C. Chevalley and R.D. Schafer, and  $E_7$  and  $E_8$  were given by H. Freudenthal as Lie algebras.

However, as Lie groups, the concrete construction of type  $E_6$ ,  $E_7$  and  $E_8$  remained unknown for the first century after classification. In [15], a simply connected compact exceptional Lie group of type  $E_6$  was constructed using algebraic techniques by I. Yokota. Later, Yokota and his fellow researchers constructed the  $E_7$  and  $E_8$  compact Lie groups and non-compact Lie groups. Yokota made an algebraic construction using Cayley algebra, split Cayley algebra and its complexification in the realization of exceptional Lie groups. Yokota was also deeply inspired by Freudenthal's treatise [3]. Yokota then developed his own method for investigating exceptional Lie groups. The Yokota-style construction method is very effective in constructing exceptional Lie groups in particular, and thus is expected to be used in the future.

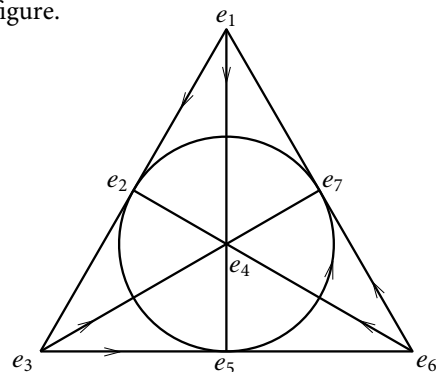
In this paper, we consider the arrangement of the subgroups in  $F_4^C$  using the Yokota-style method.

## 2. Cayley algebra and the Lie group $G_2$

### 2.1 The definition of Cayley algebra

We define a Cayley algebra, a split Cayley algebra and their complexifications and summarize their properties. First, we define a Cayley algebra.

**Definition 2.1** Let  $\mathbb{C}$  be the 8-dimensional vector space on  $\mathbf{R}$  with the basis  $1, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ . With  $e_0 = 1$  as the identity element, other products are defined as the following figure.



In the above figure, the product is defined as follows

between  $e_1, e_2, e_3$  on the line:

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_2e_3 &= -e_3e_2 = e_1, \\ e_3e_1 &= -e_1e_3 = e_2, \\ e_1e_2 &= -e_2e_1 = e_3. \end{aligned}$$

The product is defined in the same way on the other six lines. For example,  $e_5e_7 = e_2$ . Furthermore, the product is defined so that the distributive law holds for any element. The algebra  $\mathfrak{C}$  defined in this way is called the Cayley algebra. The element of  $\mathfrak{C}$  is called an octonion or a Cayley number. The Cayley algebra  $\mathfrak{C}$  is a non-associative algebra.

The product of the Cayley algebras is tabulated as follows.

	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	-1	$e_3$	$-e_2$	$e_5$	$-e_4$	$e_7$	$-e_6$
$e_2$	$-e_3$	-1	$e_1$	$-e_6$	$e_7$	$e_4$	$-e_5$
$e_3$	$e_2$	$-e_1$	-1	$e_7$	$e_6$	$-e_5$	$-e_4$
$e_4$	$-e_5$	$e_6$	$-e_7$	-1	$e_1$	$-e_2$	$e_3$
$e_5$	$e_4$	$-e_7$	$-e_6$	$-e_1$	-1	$e_3$	$e_2$
$e_6$	$-e_7$	$-e_4$	$e_5$	$e_2$	$-e_3$	-1	$e_1$
$e_7$	$e_6$	$e_5$	$e_4$	$-e_3$	$-e_2$	$-e_1$	-1

For  $a = a_0 + \sum_{i=1}^7 a_i e_i$ ,  $b = b_0 + \sum_{i=1}^7 b_i e_i$  in  $\mathfrak{C}$ , we define a conjugation  $\bar{a}$ , an inner product  $(a, b)$ , a norm  $N(a)$ , a length  $|a|$ , an  $\mathbf{R}$ -linear map  $\gamma : \mathfrak{C} \rightarrow \mathfrak{C}$  as follows:

$$\begin{aligned} \bar{a} &= a_0 - \sum_{i=1}^7 a_i, \\ (a, b) &= \sum_{i=0}^7 a_i b_i, \\ N(a) &= (a, a) = a\bar{a}, \\ |a| &= \sqrt{(a, a)}, \\ \gamma(a) &= \sum_{i=0}^3 a_i e_i - \sum_{i=4}^7 a_i e_i. \end{aligned}$$

Then it holds  $(a, b) = \frac{1}{2}(a\bar{b} + b\bar{a})$ . For a non-zero Cayley number  $a$ , we put  $\frac{\bar{a}}{|a|^2}$  as  $a^{-1}$ . Then it holds  $aa^{-1} = a^{-1}a = 1$ .

Hence  $\mathfrak{C}$  is a non-associative skew field. And, from  $\gamma^2 = 1$ ,  $\gamma$  is an involution.

The explicit forms of the products  $a\bar{a}$ ,  $a^2$ ,  $\bar{a}^2$  and  $ab$  are as follows:

$$a\bar{a} = \bar{a}a = a_0^2 + a_1^2 + \cdots + a_7^2,$$

$$\begin{aligned} a^2 &= a_0^2 - a_1^2 - \cdots - a_7^2 \\ &\quad + 2a_0a_1e_1 + 2a_0a_2e_2 + \cdots + 2a_0a_7e_7, \end{aligned}$$

$$\begin{aligned} \bar{a}^2 &= a_0^2 - a_1^2 - \cdots - a_7^2 \\ &\quad - 2a_0a_1e_1 - 2a_0a_2e_2 - \cdots - 2a_0a_7e_7, \end{aligned}$$

$$\begin{aligned} ab &= (a_0 + a_1e_1 + \cdots + a_7e_7)(b_0 + b_1e_1 + \cdots + b_7e_7) \\ &= a_0b_0 - a_1b_1 - \cdots - a_7b_7 \\ &\quad + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_4b_5 - a_5b_4 + a_6b_7 - a_7b_6)e_1 \\ &\quad + (a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3 + a_6b_4 - a_4b_6 + a_5b_7 - a_7b_5)e_2 \\ &\quad + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1 + a_4b_7 - a_7b_4 + a_5b_6 - a_6b_5)e_3 \\ &\quad + (a_0b_4 + a_4b_0 + a_5b_1 - a_1b_5 + a_2b_6 - a_6b_2 + a_7b_3 - a_3b_7)e_4 \\ &\quad + (a_0b_5 + a_5b_0 + a_1b_4 - a_4b_1 + a_7b_2 - a_2b_7 + a_6b_3 - a_3b_6)e_5 \\ &\quad + (a_0b_6 + a_6b_0 + a_7b_1 - a_1b_7 + a_4b_2 - a_2b_4 + a_3b_5 - a_5b_3)e_6 \\ &\quad + (a_0b_7 + a_7b_0 + a_1b_6 - a_6b_1 + a_2b_5 - a_5b_2 + a_3b_4 - a_4b_3)e_7. \end{aligned}$$

Next we define the split Cayley algebra.

**Definition 2.2** Let  $\mathfrak{C}'$  be the 8-dimensional vector space on  $\mathbf{R}$  with the basis  $1, e_1, e_2, e_3, e'_4, e'_5, e'_6, e'_7$ . With  $e_0 = 1$  as the identity element, other products are defined as the following table.

	$e_1$	$e_2$	$e_3$	$e'_4$	$e'_5$	$e'_6$	$e'_7$
$e_1$	-1	$e_3$	$-e_2$	$e'_5$	$-e'_4$	$-e'_7$	$e'_6$
$e_2$	$-e_3$	-1	$e_1$	$e'_6$	$e'_7$	$-e'_4$	$-e'_5$
$e_3$	$e_2$	$-e_1$	-1	$e'_7$	$-e'_6$	$e'_5$	$-e'_4$
$e'_4$	$-e'_5$	$-e'_6$	$-e'_7$	1	$-e_1$	$-e_2$	$-e_3$
$e'_5$	$e'_4$	$-e'_7$	$e'_6$	$e_1$	1	$e_3$	$-e_2$
$e'_6$	$e'_7$	$e'_4$	$-e'_5$	$e_2$	$-e_3$	1	$e_1$
$e'_7$	$-e'_6$	$e'_5$	$e'_4$	$e_3$	$e_2$	$-e_1$	1

The algebra  $\mathfrak{C}'$  defined in this way is called the split Cayley algebra.  $\mathfrak{C}'$  is a non-associative algebra.

$$\text{For } a = a_0 + \sum_{i=1}^3 a_i e_i + \sum_{i=4}^7 a_i e'_i, \quad b = b_0 + \sum_{i=1}^3 b_i e_i + \sum_{i=4}^7 b_i e'_i$$

in  $\mathfrak{C}'$ , we define a conjugation  $\bar{a}$ , an inner product  $(a, b)$ , a norm  $N(a)$  as follows :

$$\begin{aligned} \bar{a} &= a_0 - \sum_{i=1}^3 a_i e_i - \sum_{i=4}^7 a_i e'_i, \\ (a, b) &= \sum_{i=0}^3 a_i b_i - \sum_{i=4}^7 a_i b_i, \\ N(a) &= (a, a) = a\bar{a} = \sum_{i=0}^3 a_i^2 - \sum_{i=4}^7 a_i^2. \end{aligned}$$

The explicit forms of the products  $a\bar{a}$ ,  $a^2$ ,  $\bar{a}^2$  and  $ab$  are as follows:

$$\begin{aligned}
a\bar{a} &= \bar{a}a = a_0^2 + a_1^2 + a_2^2 + a_3^2 - a_4^2 - a_5^2 - a_6^2 - a_7^2, a^2 \\
&= a_0^2 - a_1^2 - a_2^2 - a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \\
&\quad + 2a_0a_1e_1 + 2a_0a_2e_2 + \cdots + 2a_0a_7e_7', \\
\bar{a}^2 &= a_0^2 - a_1^2 - a_2^2 - a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \\
&\quad - 2a_0a_1e_1 - 2a_0a_2e_2 - \cdots - 2a_0a_7e_7',
\end{aligned}$$

$$\begin{aligned}
ab &= (a_0 + a_1e_1 + \cdots + a_7e_7')(b_0 + b_1e_1 + \cdots + b_7e_7') \\
&= a_0b_0 - a_1b_1 - \cdots - a_3b_3 + a_4b_4 + \cdots + a_7b_7 \\
&\quad + (a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 + a_5b_4 - a_4b_5 + a_6b_7 - a_7b_6)e_1 \\
&\quad + (a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3 + a_6b_4 - a_4b_6 + a_7b_5 - a_5b_7)e_2 \\
&\quad + (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1 + a_7b_4 - a_4b_7 + a_5b_6 - a_6b_5)e_3 \\
&\quad + (a_0b_4 + a_4b_0 + a_5b_1 - a_1b_5 + a_6b_2 - a_2b_6 + a_7b_3 - a_3b_7)e_4 \\
&\quad + (a_0b_5 + a_5b_0 + a_1b_4 - a_4b_1 + a_7b_2 - a_2b_7 + a_3b_6 - a_6b_3)e_5 \\
&\quad + (a_0b_6 + a_6b_0 + a_1b_7 - a_7b_1 + a_2b_4 - a_4b_2 + a_5b_3 - a_3b_5)e_6 \\
&\quad + (a_0b_7 + a_7b_0 + a_6b_1 - a_1b_6 + a_2b_5 - a_5b_2 + a_3b_4 - a_4b_3)e_7'.
\end{aligned}$$

The complexification  $\mathfrak{C}^C = \{a + ib \mid a, b \in \mathfrak{C}\}$  of the Cayley algebra  $\mathfrak{C}$  is called the complex Cayley algebra. In  $\mathfrak{C}^C$ , we define a conjugation  $\bar{x}$ , an inner product  $(x, y)$ , a norm  $N(x)$ , a conjugation  $\tau(x)$  for complexification and an involution  $\gamma$  as follows :

$$\begin{aligned}
\bar{x} &= \bar{a} + i\bar{b}, \\
(x, y) &= (a, c) - (b, d) + i((a, d) + (b, c)), \\
N(x) &= x\bar{x} = a\bar{a} - b\bar{b} + i(a\bar{b} + b\bar{a}), \\
\tau(x) &= a - ib, \\
\gamma(x) &= \gamma(a) + i\gamma(b), \\
(x = a + ib, y = c + id \in \mathfrak{C}^C).
\end{aligned}$$

Then it holds  $\tau\gamma = \gamma\tau$ . In  $\mathfrak{C}^C$ , there are two complex conjugations  $\bar{x}$  and  $\tau(x)$ . The explicit forms of these are as follows:

$$\begin{aligned}
\bar{x} &= a_0 - a_1e_1 - \cdots - a_7e_7 + i(b_0 - b_1e_1 - \cdots - b_7e_7), \\
(x, y) &= a_0c_0 + \cdots + a_7c_7 - b_0d_0 - \cdots - b_7d_7 \\
&\quad + i(a_0d_0 + \cdots + a_7d_7 + b_0c_0 + \cdots + b_7c_7), \\
N(x) &= a_0^2 + \cdots + a_7^2 - b_0^2 - \cdots - b_7^2 + i(2a_0b_0 + \cdots + 2a_7b_7) \\
&= (a_0 + ib_0)^2 + (a_1 + ib_1)^2 + \cdots + (a_7 + ib_7)^2, \\
\tau(x) &= a_0 + a_1e_1 + \cdots + a_7e_7 - i(b_0 + b_1e_1 + \cdots + b_7e_7), \\
\gamma(x) &= a_0 + \cdots + a_3e_3 - a_4e_4 - \cdots - a_7e_7 \\
&\quad + i(b_0 + \cdots + b_3e_3 - b_4e_4 - \cdots - b_7e_7), \\
\tau\gamma(x) &= a_0 + \cdots + a_3e_3 - a_4e_4 - \cdots - a_7e_7 \\
&\quad + i(-b_0 - \cdots - b_3e_3 + b_4e_4 + \cdots + b_7e_7).
\end{aligned}$$

Let  $\varphi$  be the linear map from  $\mathfrak{C}'$  to  $\mathfrak{C}^C$  that corresponds

1,  $e_1, e_2, e_3, e_4', e_5', e_6', e_7'$  to 1,  $e_1, e_2, e_3, ie_4, ie_5, -ie_6, ie_7$ , respectively. Then  $\varphi$  is an injective homomorphism as algebras. Therefore, the split Cayley algebra  $\mathfrak{C}'$  is isomorphic to the subalgebra  $\langle 1, e_1, e_2, e_3, ie_4, ie_5, -ie_6, ie_7 \rangle_{\mathbf{R}}$  of  $\mathfrak{C}^C$ :

$$\mathfrak{C}' \subseteq \mathfrak{C}^C.$$

**Proposition 2.3** The complexifications of the Cayley algebra  $\mathfrak{C}$  and the split Cayley algebra  $\mathfrak{C}'$  are isomorphic:

$$\mathfrak{C}^C \cong \mathfrak{C}'^C.$$

(Proof.) The map from  $\mathfrak{C}'^C$  to  $\mathfrak{C}^C$  that makes  $a + ib \in \mathfrak{C}'^C$  correspond to  $\varphi(a) + i\varphi(b)$  is an isomorphism.  $\square$

$\mathfrak{C}$  doesn't have an associative law, but it replaces the following formulas in  $\mathfrak{C}$ :

1.  $\overline{ab} = \bar{b}\bar{a}$ .
2.  $(aa)b = a(ab)$ ,  $(ab)a = a(ba)$ ,  $b(aa) = (ba)a$ .  
 $(a\bar{a})b = a(\bar{a}b)$ ,  $(ab)\bar{a} = a(b\bar{a})$ ,  $b(a\bar{a}) = (ba)\bar{a}$ .
3.  $(ab)c + b(ca) = a(bc) + (bc)a$ ,  
 $(ab)c + (ac)b = a(bc) + a(cb)$ ,  
 $(ab)c + (ba)c = a(bc) + b(ac)$ .
4.  $(ab)(ca) = a(bc)a$ , (Moufang's formula)  
 $(ab)(\bar{b}\bar{a}) = a(\bar{b}\bar{b})\bar{a} = (a, a)(b, b)$ .
5.  $(a, a) = a\bar{a} = \bar{a}a$ ,  
 $(a, b) = \frac{1}{2}(a\bar{b} + b\bar{a}) = \frac{1}{2}(\bar{a}b + \bar{b}a)$ ,  
 $(a, b)c = \frac{1}{2}((ca)\bar{b} + (cb)\bar{a}) = \frac{1}{2}(\bar{a}(bc) + \bar{b}(ac))$ .
6.  $(a, b) = (b, a) = (\bar{a}, \bar{b}) = (\bar{b}, \bar{a})$ .
7.  $(ab, ab) = (a, a)(b, b)$ ,  
 $(ab, ac) = (a, a)(b, c) = (ba, ca)$ ,  
 $(a, b)(c, d) = \frac{1}{2}((ac, bd) + (ad, bc))$ .
8.  $(ab, c) = (b, \bar{a}c)$ ,  $(ba, c) = (b, \bar{c}a)$ .
9. When  $u_0 = 1, u_1, u_2, \dots, u_m$  are the normal orthonormal basis.  
 $u_i(u_j a) = -u_j(u_i a)$ , ( $i \neq j$ ). especially,  $u_i u_j = -u_j u_i$ .  
 $u_i(u_i a) = -a$ . especially,  $u_i^2 = -1$ .  
 $u_i(u_j u_k) = u_j(u_k u_i) = u_k(u_i u_j)$ , ( $i, j, k$  are different).

## 2.2 Exceptional Lie group $G_2$

We define the Lie groups of type  $G_2$  and investigate their properties.

**Definition 2.4** We define groups  $G_2, G_{2(2)}$  and  $G_2^C$  as automorphism groups of Jordan algebras :

$$G_2 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}) \mid \alpha(xy) = \alpha(x)\alpha(y)\},$$

$$G_{2(2)} = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{C}') \mid \alpha(xy) = \alpha(x)\alpha(y)\},$$

$$G_2^C = \{ \alpha \in \text{Iso}_{\mathbb{C}}(\mathbb{C}^C) \mid \alpha(xy) = \alpha(x)\alpha(y) \}.$$

Then  $G_2$ ,  $G_{2(2)}$  and  $G_2^C$  are compact, non-compact and complex Lie groups of type  $G_2$ , respectively.

For  $\alpha \in G_2$ , we can define the complexified map  $\alpha^C : \mathbb{C}^C \rightarrow \mathbb{C}^C$  as  $\alpha^C(a + ib) = \alpha(a) + i\alpha(b)$ , where  $a, b \in \mathbb{C}$ . Then we identify  $\alpha \in G_2$  with  $\alpha^C \in G_2^C$  :

$$G_2 \subseteq G_2^C.$$

Similarly, from  $\mathbb{C}'^C = \mathbb{C}^C$ , for  $\alpha \in G_{2(2)}$ , we can define the complexified map  $\alpha^C : \mathbb{C}^C \rightarrow \mathbb{C}^C$  as  $\alpha^C(a + ib) = \alpha(a) + i\alpha(b)$ , where  $a, b \in \mathbb{C}'$ . Then we identify  $\alpha \in G_{2(2)}$  with  $\alpha^C \in G_2^C$  :

$$G_{2(2)} \subseteq G_2^C.$$

We define  $(G_2^C)^\tau$  and  $(G_2^C)^{\tau\gamma}$  as follows :

$$(G_2^C)^\tau = \{ \alpha \in G_2^C \mid \alpha\tau = \tau\alpha \},$$

$$(G_2^C)^{\tau\gamma} = \{ \alpha \in G_2^C \mid \alpha\tau\gamma = \tau\gamma\alpha \}.$$

Then  $G_2$  and  $G_{2(2)}$  are isomorphic to  $(G_2^C)^\tau$  and  $(G_2^C)^{\tau\gamma}$ , respectively:

$$G_2 = (G_2^C)^\tau \subseteq G_2^C,$$

$$G_{2(2)} = (G_2^C)^{\tau\gamma} \subseteq G_2^C.$$

At this time,  $\gamma$  is the element of  $G_2$  and  $G_{2(2)}$ :

$$\gamma \in G_2 \cap G_{2(2)} \subseteq G_2^C.$$

In general, for a group  $G$  and an involution  $\mu$ , we define  $G^\mu$  as follows:

$$G^\mu = \{ g \in G \mid g\mu = \mu g \}.$$

### 2.3 Other $\gamma$ type involutions

We consider other  $\gamma$  type involutions. Involutions  $\gamma', \gamma''$  of type  $\gamma$  are defined as follows:

$$\gamma'(a) = a_0 + a_1e_1 - a_2e_2 - a_3e_3 + a_4e_4 + a_5e_5 - a_6e_6 - a_7e_7,$$

$$\gamma''(a) = a_0 - a_1e_1 + a_2e_2 - a_3e_3 + a_4e_4 - a_5e_5 + a_6e_6 - a_7e_7.$$

By expressing  $a \in \mathbb{C}$  as  $a = m + ne_4 \in \mathbf{H} \oplus \mathbf{H}e_4$ ,

$$(m, n \in \mathbf{H} = \mathbf{C} \oplus \mathbf{C}e_2, \text{ where } \mathbf{C} = \mathbf{R} \oplus \mathbf{R}e_1)$$

$\gamma'$  and  $\gamma''$  can also be defined as

$$\gamma'(a) = \gamma'(m) + \gamma'(n)e_4,$$

$$\gamma''(a) = \gamma''(m) + \gamma''(n)e_4,$$

where  $\gamma'(m_1 + m_2e_2) = m_1 - m_2e_2$ ,

$$\gamma''(m_1 + m_2e_2) = \overline{m_1} + \overline{m_1}e_2.$$

for  $m = m_1 + m_2e_2 \in \mathbf{H} = \mathbf{C} \oplus \mathbf{C}e_2$ . The same applies to  $n$ .

Therefore, as notations, we often write

$$\gamma = \gamma_{\mathbb{C}} = \gamma_{123}, \quad \gamma' = \gamma_{\mathbf{H}} = \gamma_{145}, \quad \gamma'' = \gamma_{\mathbf{C}} = \gamma_{642}.$$

Then,  $\gamma^2 = \gamma'^2 = \gamma''^2 = 1$  and  $\gamma, \gamma', \gamma''$  are commutative, respectively:

$$\gamma\gamma' = \gamma'\gamma, \quad \gamma\gamma'' = \gamma''\gamma, \quad \gamma'\gamma'' = \gamma''\gamma'.$$

From the definition of  $\gamma, \gamma', \gamma''$ , we have

$$\gamma, \gamma', \gamma'' \in G_2.$$

Since the algebra generated by

$$1, e_1, e_4, e_5, ie_6, ie_7, ie_2, ie_3$$

is isomorphic to the split Cayley algebra  $\mathbb{C}'$ , as in the case of  $\gamma$ , we can get

$$(G_2^C)^{\tau\gamma} \cong G_{2(2)},$$

where  $\gamma' = \gamma_{145}$ . Similarly, since the algebra generated by

$$1, e_6, e_4, e_2, ie_3, ie_5, ie_7, ie_1$$

is isomorphic to the split Cayley algebra  $\mathbb{C}'$ , we can get

$$(G_2^C)^{\tau\gamma''} \cong G_{2(2)},$$

where  $\gamma'' = \gamma_{642}$ . We put  $\gamma\gamma'$  as  $\gamma'''$ :

$$\gamma''' = \gamma\gamma'.$$

Then we have

$$\gamma\gamma' = \gamma''', \quad \gamma'\gamma''' = \gamma, \quad \gamma'''\gamma = \gamma'.$$

In this paper, in order to investigate the relation between the arrangement of subgroups of Lie groups and involutions, this relational expression is clarified as a concept.

**Definition 2.5** For a group  $G$  and an ordered set  $\{\mu_1, \mu_2, \mu_3\}$  of automorphisms of  $G$ , if they satisfy the following conditions

$$\mu_1\mu_2 = \mu_3, \quad \mu_2\mu_3 = \mu_1, \quad \mu_3\mu_1 = \mu_2,$$

then, we call them cyclic. Moreover, if any two automorphisms is commutative, they are called commutative cyclic automorphisms. When there is no confusion, we omit the parentheses.

For cyclic automorphisms  $\mu_1, \mu_2, \mu_3$  of  $G$ , each  $G^{\mu_i}$  is a subgroup of  $G$ . So, we can write the following diagram.

$$\begin{array}{ccc}
& G & \\
\nearrow & \uparrow & \nwarrow \\
G^{\mu_1} & G^{\mu_2} & G^{\mu_3}
\end{array}$$

Then, each two intersections  $G^{\mu_i} \cap G^{\mu_j}$  is equal to the three intersections:

$$G^{\mu_1} \cap G^{\mu_2} = G^{\mu_2} \cap G^{\mu_3} = G^{\mu_3} \cap G^{\mu_1} = G^{\mu_1} \cap G^{\mu_2} \cap G^{\mu_3}.$$

From now on, we will write  $G^{\mu_i} \cap G^{\mu_j}$  as  $G^{\mu_i, \mu_j}$  and  $G^{\mu_i} \cap G^{\mu_j} \cap G^{\mu_k}$  as  $G^{\mu_i, \mu_j, \mu_k}$ . Using this notation, the above relational expression can be written as:

$$G^{\mu_1, \mu_2} = G^{\mu_2, \mu_3} = G^{\mu_3, \mu_1} = G^{\mu_1, \mu_2, \mu_3}.$$

**Proposition 2.6** For involutions  $\gamma = \gamma_{\mathbb{C}}$ ,  $\gamma' = \gamma_{\mathbb{H}}$ ,  $\gamma'' = \gamma_{\mathbb{C}}$  of  $G_2^{\mathbb{C}}$ ,

$$\{\gamma, \gamma', \gamma\gamma'\}, \{\gamma, \gamma'', \gamma\gamma''\}, \{\gamma', \gamma'', \gamma'\gamma''\}$$

are commutative cyclic involutions, respectively.

(Proof.) These are obtained by direct calculations.  $\square$

By using  $\tau$ , we get the following diagram.

$$\begin{array}{ccc}
& G_2^{\mathbb{C}} & \\
\nearrow & \uparrow & \nwarrow \\
(G_2^{\mathbb{C}})^{\tau\gamma} & (G_2^{\mathbb{C}})^{\tau} & (G_2^{\mathbb{C}})^{\tau\gamma'}
\end{array}$$

This diagram means

$$\begin{array}{ccc}
& G_2^{\mathbb{C}} & \\
\nearrow & \uparrow & \nwarrow \\
G_{2(2)} & G_2 & G_{2(2)}
\end{array}$$

### 3. Jordan algebra and the Lie group $F_4$

#### 3.1 Jordan algebra

First, we define Jordan algebra. Let  $\mathfrak{J} = \mathfrak{J}(3, \mathbb{C})$  denote all  $3 \times 3$  Hermitian matrices with entries in the Cayley algebra  $\mathbb{C}$ . Any element  $X \in \mathfrak{J}$  is of the form

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix},$$

where  $\xi_i \in \mathbf{R}$ ,  $x_i \in \mathbb{C}$ . In  $\mathfrak{J}$ , the multiplication  $X \circ Y$  is defined by

$$X \circ Y = \frac{1}{2}(XY + YX)$$

which is called the Jordan multiplication. Then  $\mathfrak{J}$  is called Jordan algebra. We define a trace  $\text{tr}(X)$ , an inner product  $(X, Y)$  and a trilinear form  $\text{tr}(X, Y, Z)$  respectively by

$$\text{tr}(X) = \xi_1 + \xi_2 + \xi_3, \quad X = X(\xi, x)$$

$$(X, Y) = \text{tr}(X \circ Y),$$

$$\text{tr}(X, Y, Z) = (X, Y \circ Z),$$

Moreover we define the Freudenthal multiplication  $X \times Y$  by

$$X \times Y = \frac{1}{2}(2X \circ Y - \text{tr}(X)Y - \text{tr}(Y)X + (\text{tr}(X)\text{tr}(Y) - (X, Y))E),$$

where  $E$  is the  $3 \times 3$  unit matrix, and we define a trilinear form  $(X, Y, Z)$  and a determinant  $\det X$  respectively by

$$(X, Y, Z) = (X, Y \times Z),$$

$$\det X = \frac{1}{3}(X, X, X).$$

For  $X = X(\xi, x)$ ,  $Y = Y(\eta, y)$  and  $Z = Z(\zeta, z) \in \mathfrak{J}$ , the explicit forms in the terms of their entries are as follows.

$$(X, Y) = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 + 2((x_1, y_1) + (x_2, y_2) + (x_3, y_3)),$$

$$\text{tr}(X, Y, Z) = \xi_1\eta_1\zeta_1 + \xi_2\eta_2\zeta_2 + \xi_3\eta_3\zeta_3 + R(x_1y_2z_3 + x_1z_2y_3)$$

$$+ R(x_2y_3z_1 + x_2z_3y_1) + R(x_3y_1z_2 + x_3z_1y_2)$$

$$+ \xi_1((y_2, z_2) + (y_3, z_3)) + \xi_2((y_3, z_3) + (y_1, z_1))$$

$$+ \xi_3((y_1, z_1) + (y_2, z_2))$$

$$+ \eta_1((z_2, x_2) + (z_3, x_3)) + \eta_2((z_3, x_3) + (z_1, x_1))$$

$$+ \eta_3((z_1, x_1) + (z_2, x_2))$$

$$+ \zeta_1(x_2, y_2) + (x_3, y_3) + \zeta_2((x_3, y_3) + (x_1, y_1))$$

$$+ \zeta_3((x_1, y_1) + (x_2, y_2)),$$

$$(X, Y, Z) = \frac{1}{2}(\xi_1\eta_2\zeta_3 + \xi_1\eta_3\zeta_2 + \xi_2\eta_3\zeta_1 + \xi_2\eta_1\zeta_3$$

$$+ \xi_3\eta_1\zeta_2 + \xi_3\eta_2\zeta_1)$$

$$+ R(x_1y_2z_3 + x_1z_2y_3) + R(x_2y_3z_1 + x_2z_3y_1)$$

$$+ R(x_3y_1z_2 + x_3z_1y_2)$$

$$- \xi_1(y_1, z_1) - \xi_2(y_2, z_2) - \xi_3(y_3, z_3)$$

$$- \eta_1(z_1, x_1) - \eta_2(z_2, x_2) - \eta_3(z_3, x_3)$$

$$- \zeta_1(x_1, y_1) - \zeta_2(x_2, y_2) - \zeta_3(x_3, y_3),$$

$$\det X = \xi_1\xi_2\xi_3 + 2R(x_1x_2x_3)$$

$$- \xi_1x_1\overline{x_1} - \xi_2x_2\overline{x_2} - \xi_3x_3\overline{x_3},$$

where  $R(x)$  denotes the real part of  $x \in \mathbb{C}$ .

For  $X = X(\xi, x)$  and  $Y = Y(\eta, y) \in \mathfrak{J}$ , the explicit form of  $X \circ Y$  is as follows.

$$X \circ Y = \frac{1}{2} \begin{pmatrix} \zeta_1 & z_3 & \overline{z_2} \\ \overline{z_3} & \zeta_2 & z_1 \\ z_2 & \overline{z_1} & \zeta_3 \end{pmatrix},$$

$$\zeta_1 = 2\xi_1\eta_1 + x_3\overline{y_3} + y_3\overline{x_3} + \overline{x_2}y_2 + \overline{y_2}x_2,$$

$$\zeta_2 = 2\xi_2\eta_2 + x_1\overline{y_1} + y_1\overline{x_1} + \overline{x_3}y_3 + \overline{y_3}x_3,$$

$$\zeta_3 = 2\xi_3\eta_3 + x_2\overline{y_2} + y_2\overline{x_2} + \overline{x_1}y_1 + \overline{y_1}x_1,$$

$$z_1 = (\eta_2 + \eta_3)x_1 + (\xi_2 + \xi_3)y_1 + \overline{x_2}y_3 + \overline{y_2}x_3,$$

$$z_2 = (\eta_3 + \eta_1)x_2 + (\xi_3 + \xi_1)y_2 + \overline{x_3}y_1 + \overline{y_3}x_1,$$

$$z_3 = (\eta_1 + \eta_2)x_3 + (\xi_1 + \xi_2)y_3 + \overline{x_1}y_2 + \overline{y_1}x_2.$$

Especially,  $X \circ X$  is as follows.

$$X \circ X = X^2 = \begin{pmatrix} \zeta_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & \zeta_2 & z_1 \\ z_2 & \bar{z}_1 & \zeta_3 \end{pmatrix},$$

$$\zeta_1 = \xi_1^2 + x_3 \bar{x}_3 + \bar{x}_2 x_2,$$

$$\zeta_2 = \xi_2^2 + x_1 \bar{x}_1 + \bar{x}_3 x_3,$$

$$\zeta_3 = \xi_3^2 + x_2 \bar{x}_2 + \bar{x}_1 x_1,$$

$$z_1 = (\xi_2 + \xi_3)x_1 + \bar{x}_2 \bar{x}_3,$$

$$z_2 = (\xi_3 + \xi_1)x_2 + \bar{x}_3 \bar{x}_1,$$

$$z_3 = (\xi_1 + \xi_2)x_3 + \bar{x}_1 \bar{x}_2.$$

For  $X = X(\xi, x)$  and  $Y = Y(\eta, y) \in \mathfrak{J}$ , the explicit form of  $X \times Y$  is as follows.

$$X \times Y = \frac{1}{2} \begin{pmatrix} \zeta_1 & z_3 & \bar{z}_2 \\ \bar{z}_3 & \zeta_2 & z_1 \\ z_2 & \bar{z}_1 & \zeta_3 \end{pmatrix},$$

$$\zeta_1 = \xi_2 \eta_3 + \xi_3 \eta_2 - (x_1 \bar{y}_1 + y_1 \bar{x}_1),$$

$$\zeta_2 = \xi_3 \eta_1 + \xi_1 \eta_3 - (x_2 \bar{y}_2 + y_2 \bar{x}_2),$$

$$\zeta_3 = \xi_1 \eta_2 + \xi_2 \eta_1 - (x_3 \bar{y}_3 + y_3 \bar{x}_3),$$

$$z_1 = \bar{x}_2 \bar{y}_3 + y_2 \bar{x}_3 - \xi_1 y_1 - \eta_1 x_1,$$

$$z_2 = \bar{x}_3 \bar{y}_1 + y_3 \bar{x}_1 - \xi_2 y_2 - \eta_2 x_2,$$

$$z_3 = \bar{x}_1 \bar{y}_2 + y_1 \bar{x}_2 - \xi_3 y_3 - \eta_3 x_3.$$

Especially,  $X \times X$  is as follows.

$$\begin{pmatrix} \xi_2 \xi_3 - x_1 \bar{x}_1 & \bar{x}_1 \bar{x}_2 - \xi_3 x_3 & x_3 x_1 - \xi_2 \bar{x}_2 \\ x_1 x_2 - \xi_3 \bar{x}_3 & \xi_3 \xi_1 - x_2 \bar{x}_2 & \bar{x}_2 \bar{x}_3 - \xi_1 x_1 \\ \bar{x}_3 \bar{x}_1 - \xi_2 x_2 & x_2 x_3 - \xi_1 \bar{x}_1 & \xi_1 \xi_2 - x_3 \bar{x}_3 \end{pmatrix}.$$

**Lemma 3.1** The followings hold in  $\mathfrak{J}$ .

$$(1) X \circ Y = Y \circ X, \quad X \times Y = Y \times X.$$

$$(2) E \circ E = E, \quad E \times E = E.$$

$$E \circ X = X, \quad E \times X = \frac{1}{2}(\text{tr}(X)E - X).$$

(3) The inner product  $(X, Y)$  is symmetric and positive definite.

$$(4) \text{tr}(X, Y, Z) = \text{tr}(Y, Z, X) = \text{tr}(Z, X, Y) \\ = \text{tr}(X, Z, Y) = \text{tr}(Y, X, Z) = \text{tr}(Z, Y, X).$$

The similar statement is also valid for  $(X, Y, Z)$ .

$$(5) (X, E) = (X, E, E) = \text{tr}(X, E, E) = \text{tr}(X),$$

$$\text{tr}(X, Y, E) = (X, Y).$$

$$(6) \text{tr}(X \times Y) = \frac{1}{2}(\text{tr}(X)\text{tr}(Y) - (X, Y)).$$

$$(7) (X \times X) \circ X = (\det X)E \quad (\text{Hamilton-Cayley}).$$

$$(8) (X \times X) \times (X \times X) = (\det X)X.$$

(Proof.) These are obtained by direct calculations.  $\square$

In  $\mathfrak{J}$ , we use the following notations:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix}, \\ F_2(x) = \begin{pmatrix} 0 & 0 & \bar{x} \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix}, \quad F_3(x) = \begin{pmatrix} 0 & x & 0 \\ \bar{x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the Jordan multiplication and the Freudenthal multiplication, we have the following formulas:

$$1. E_i \circ E_i = E_i, \quad E_i \circ E_j = 0, \quad (i \neq j).$$

$$2. E_i \circ F_i(x) = 0, \quad E_i \circ F_j(x) = \frac{1}{2}F_j(x), \quad (i \neq j).$$

$$3. F_i(x) \circ F_i(y) = (x, y)(E_{i+1} + E_{i+2}),$$

$$F_i(x) \circ F_{i+1}(y) = \frac{1}{2}F_{i+2}(\bar{xy}).$$

$$4. E_i \times E_i = 0, \quad E_i \times E_{i+1} = \frac{1}{2}E_{i+2}.$$

$$5. E_i \times F_i(x) = -F_i(x), \quad E_i \times F_j(x) = 0, \quad (i \neq j).$$

$$6. F_i(x) \times F_i(y) = -(x, y)E_i,$$

$$F_i(x) \times F_{i+1}(y) = \frac{1}{2}F_{i+2}(\bar{xy}).$$

In these formulas, the indexes are considered as mod 3.

### 3.2 Complex Jordan algebra

We define the complex Jordan algebra  $\mathfrak{J}^{\mathbb{C}}$  as the complexification of the Jordan algebra  $\mathfrak{J}$ :

$$\mathfrak{J}^{\mathbb{C}} = \{ X_1 + iX_2 \mid X_1, X_2 \in \mathfrak{J} \}.$$

Any element  $X \in \mathfrak{J}^{\mathbb{C}}$  is of the form

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where  $\xi_j \in \mathbb{C} = \mathbf{R} \oplus i\mathbf{R}$ ,  $x_j \in \mathbb{C} = \mathbb{C} \oplus i\mathbb{C}$ .

Then,  $\mathfrak{J}^{\mathbb{C}}$  has two complex conjugations as follows:

$$\overline{X_1 + iX_2} = \bar{X}_1 + i\bar{X}_2,$$

$$\tau(X_1 + iX_2) = X_1 - iX_2, \quad X_j \in \mathfrak{J}.$$

For  $X = X_1 + iX_2$ ,  $Y = Y_1 + iY_2 \in \mathfrak{J}^{\mathbb{C}}$ , we define the multiplication  $X \circ Y$  and  $X \times Y$  as follows:

$$X \circ Y = X_1 \circ Y_1 - X_2 \circ Y_2 + i(X_1 \circ Y_2 + X_2 \circ Y_1),$$

$$X \times Y = X_1 \times Y_1 - X_2 \times Y_2 + i(X_1 \times Y_2 + X_2 \times Y_1).$$

$\mathfrak{J}^{\mathbb{C}}$  is called the complex exceptional Jordan algebra.

**Lemma 3.2** For  $\alpha \in \text{Iso}_{\mathbb{C}}(\mathfrak{J}^{\mathbb{C}})$ , the following three conditions are equivalent.

- (1)  $\det(\alpha X) = \det X$  for all  $X \in \mathfrak{J}^C$ .
- (2)  $(\alpha X, \alpha Y, \alpha Z) = (X, Y, Z)$  for all  $X, Y, Z \in \mathfrak{J}^C$ .
- (3)  $\alpha X \times \alpha Y = {}^t\alpha^{-1}(X \times Y)$  for all  $X, Y \in \mathfrak{J}^C$ .

(Proof.) See [16, Lemma 2.1.1.]. □

### 3.3 Split Jordan algebra

For Cayley algebra  $\mathbb{C}$  and split Cayley algebra  $\mathbb{C}'$ , we define two types of split Jordan algebras as follows :

$$\begin{aligned}\mathfrak{J}(3, \mathbb{C}') &= \{X \in M(3, \mathbb{C}') \mid X^* = X\}, \\ \mathfrak{J}(1, 2, \mathbb{C}) &= \{X \in M(3, \mathbb{C}) \mid I_1 X^* I_1 = X\},\end{aligned}$$

where  $I_1 = -E_1 + E_2 + E_3$ :

$$I_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In  $\mathfrak{J}(3, \mathbb{C}')$ , any element  $X$  is of the form

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where  $\xi_j \in \mathbf{R}$ ,  $x_j \in \mathbb{C}'$ . And in  $\mathfrak{J}(1, 2, \mathbb{C})$ , any element  $X$  is of the form

$$X = X(\xi, x) = \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where  $\xi_j \in \mathbf{R}$ ,  $x_j \in \mathbb{C}$ . In  $\mathfrak{J}(3, \mathbb{C}')$  and  $\mathfrak{J}(1, 2, \mathbb{C})$ , the Jordan multiplication  $X \circ Y$  is defined by

$$X \circ Y = \frac{1}{2}(XY + YX).$$

We define the  $(\mathfrak{J}^C)_{\tau\gamma}$  and  $(\mathfrak{J}^C)_{\tau\sigma}$  as follows :

$$\begin{aligned}(\mathfrak{J}^C)_{\tau\gamma} &= \{X \in \mathfrak{J}^C \mid \tau\gamma X = X\}, \\ (\mathfrak{J}^C)_{\tau\sigma} &= \{X \in \mathfrak{J}^C \mid \tau\sigma X = X\}.\end{aligned}$$

Then,

$$\begin{aligned}\mathfrak{J}(3, \mathbb{C}') &\cong (\mathfrak{J}^C)_{\tau\gamma}, \\ \mathfrak{J}(1, 2, \mathbb{C}) &\cong (\mathfrak{J}^C)_{\tau\sigma}.\end{aligned}$$

The correspondence between  $\mathfrak{J}(1, 2, \mathbb{C})$  and  $(\mathfrak{J}^C)_{\tau\sigma}$  as a Jordan algebra is as follows :

$$\begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ -\bar{x}_3 & \xi_2 & x_1 \\ -x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} \xi_1 & ix_3 & i\bar{x}_2 \\ i\bar{x}_3 & \xi_2 & x_1 \\ ix_2 & \bar{x}_1 & \xi_3 \end{pmatrix},$$

where  $\xi_j \in \mathbf{R}$ ,  $x_j \in \mathbb{C}$ . The complexifications of  $\mathfrak{J}(3, \mathbb{C}')$  and  $\mathfrak{J}(1, 2, \mathbb{C})$  are isomorphic to  $\mathfrak{J}^C$ , respectively. So, we can identify them:

$$\mathfrak{J}^C = \mathfrak{J}(3, \mathbb{C}')^C = \mathfrak{J}(1, 2, \mathbb{C})^C.$$

### 3.4 Exceptional Lie group $F_4$

We define the Lie group  $F_4$  and consider its involutions.

**Definition 3.3** We define the group  $F_4$  as the automorphism group of the Jordan algebra  $\mathfrak{J}$  :

$$F_4 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)\}.$$

**Theorem 3.4** We can also define  $F_4$  as follows:

$$\begin{aligned}F_4 &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \times Y) = \alpha(X) \times \alpha(Y)\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z) \\ &\quad (\alpha X, \alpha Y) = (X, Y)\}\end{aligned}$$

$$= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \det(\alpha X) = \det X, (\alpha X, \alpha Y) = (X, Y)\}$$

$$= \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \det(\alpha X) = \det X, \alpha E = E\}.$$

(Proof.) See [19, Lemma 2.2.4]. □

$F_4$  contains  $G_2$  as a subgroup in the following way.

For  $\alpha \in G_2$ , we define the mapping  $\tilde{\alpha} : \mathfrak{J} \rightarrow \mathfrak{J}$  as

$$\tilde{\alpha} \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \alpha x_3 & \overline{\alpha x_2} \\ \overline{\alpha x_3} & \xi_2 & \alpha x_1 \\ \alpha x_2 & \overline{\alpha x_1} & \xi_3 \end{pmatrix}.$$

Then  $\tilde{\alpha} \in F_4$ . So we identify  $\alpha \in G_2$  with  $\tilde{\alpha} \in F_4$  :

$$G_2 \subseteq F_4.$$

We often write the same notation  $\tilde{\alpha} = \alpha$ .

For the map  $\gamma : \mathbb{C} \rightarrow \mathbb{C}$ ,  $\mathbf{R}$ -linear map  $\gamma : \mathfrak{J} \rightarrow \mathfrak{J}$  is defined by

$$\gamma \begin{pmatrix} \xi_1 & x_3 & \bar{x}_2 \\ \bar{x}_3 & \xi_2 & x_1 \\ x_2 & \bar{x}_1 & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & \gamma x_3 & \overline{\gamma x_2} \\ \overline{\gamma x_3} & \xi_2 & \gamma x_1 \\ \gamma x_2 & \overline{\gamma x_1} & \xi_3 \end{pmatrix}.$$

Then we get  $\gamma^2 = \gamma$ . By this correspondence, we consider  $\gamma \in G_2$  to be  $\gamma \in F_4$ .

$$\gamma \in G_2 \subseteq F_4.$$

We consider the following subgroup  $F_4^\gamma$  of  $F_4$  :

$$F_4^\gamma = \{\alpha \in F_4 \mid \alpha\gamma = \gamma\alpha\}.$$

We get the following diagram.

$$\begin{array}{ccc} F_4^\gamma & \rightarrow & F_4 \\ \uparrow & & \uparrow \\ G_2^\gamma & \rightarrow & G_2 \end{array}$$

To investigate the group  $F_4^\gamma$ , we decompose  $\mathfrak{J}$  into eigenspaces :

$$\mathfrak{J} = \mathfrak{J}_\gamma \oplus \mathfrak{J}_{-\gamma},$$

where

$$\begin{aligned} \mathfrak{J}_\gamma &= \{X \in \mathfrak{J} \mid \gamma X = X\}, \\ \mathfrak{J}_{-\gamma} &= \{X \in \mathfrak{J} \mid \gamma X = -X\}. \end{aligned}$$

Any element  $X \in \mathfrak{J}_\gamma$  is of the form

$$X = \begin{pmatrix} \xi_1 & a_3 & \overline{a_2} \\ \overline{a_3} & \xi_2 & a_1 \\ a_2 & \overline{a_1} & \xi_3 \end{pmatrix},$$

where  $\xi_i \in \mathbf{R}$ ,  $a_j \in \mathbf{H}$ . And any element  $X \in \mathfrak{J}_{-\gamma}$  is of the form

$$X = \begin{pmatrix} 0 & a_3 e_4 & -a_2 e_4 \\ -a_3 e_4 & 0 & a_1 e_4 \\ a_2 e_4 & -a_1 e_4 & 0 \end{pmatrix},$$

where  $a_j \in \mathbf{H}$ .

We define the  $\mathbf{R}$ -linear map  $\sigma : \mathfrak{J} \rightarrow \mathfrak{J}$  as

$$\sigma \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & -\overline{x_2} \\ -\overline{x_3} & \xi_2 & x_1 \\ -x_2 & \overline{x_1} & \xi_3 \end{pmatrix}.$$

Then we get  $\sigma \in F_4$  and  $\sigma^2 = \sigma$ . We consider the following subgroup  $F_4^\sigma$  of  $F_4$ :

$$F_4^\sigma = \{\alpha \in F_4 \mid \alpha \sigma = \sigma \alpha\}.$$

To investigate the group  $F_4^\sigma$ , we decompose  $\mathfrak{J}$  into eigenspaces :

$$\mathfrak{J} = \mathfrak{J}_\sigma \oplus \mathfrak{J}_{-\sigma},$$

where

$$\begin{aligned} \mathfrak{J}_\sigma &= \{X \in \mathfrak{J} \mid \sigma X = X\}, \\ \mathfrak{J}_{-\sigma} &= \{X \in \mathfrak{J} \mid \sigma X = -X\}. \end{aligned}$$

For  $X = X(\xi, x)$ , we get

$$E_1 \circ X = \frac{1}{2} \begin{pmatrix} 2\xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}.$$

Hence, the following holds:

$$\mathfrak{J}_\sigma = \{X \in \mathfrak{J} \mid E_1 \circ X = \xi E_1, \xi \in \mathbf{R}\},$$

$$\mathfrak{J}_{-\sigma} = \{X \in \mathfrak{J} \mid E_1 \circ X = \frac{1}{2} X\},$$

where any element  $X \in \mathfrak{J}_\sigma$  and  $Y \in \mathfrak{J}_{-\sigma}$  are of the form

$$X = \begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \overline{x_1} & \xi_3 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & 0 \\ x_2 & 0 & 0 \end{pmatrix}.$$

Spin(8) and Spin(9) are realized in  $F_4$  as follows :

$$\text{Spin}(8) = \{\alpha \in F_4 \mid \alpha E_i = E_i, i = 1, 2, 3\},$$

$$\text{Spin}(9) = (F_4)_{E_1} = \{\alpha \in F_4 \mid \alpha E_1 = E_1\}.$$

By using eigendecomposition by  $\sigma$ , we get

$$F_4^\sigma = (F_4)_{E_1} \cong \text{Spin}(9).$$

Hence we have

$$G_2 \subseteq \text{Spin}(8) \subseteq \text{Spin}(9) \subseteq F_4.$$

### 3.5 Cayley projective plane

We define Cayley projective plane  $\mathbb{C}P_2$  as

$$\mathbb{C}P_2 = \{X \in \mathfrak{J} \mid X^2 = X, \text{tr}(X) = 1\}.$$

We often refer to  $\mathbb{C}P_2$  simply as Cayley plane.

**Theorem 3.5**

$$\mathbb{C}P_2 \cong F_4/\text{Spin}(9)$$

(Proof.) For  $\alpha \in F_4$  and  $X \in \mathbb{C}P_2$ , we have  $\alpha X \in \mathbb{C}P_2$ . Hence the group  $F_4$  acts on  $\mathbb{C}P_2$ . Then this action is transitive. And the isotropy subgroup of  $F_4$  at  $E_1$  is  $(F_4)_{E_1} = \text{Spin}(9)$ .  $\square$

From  $F_4^\sigma = (F_4)_{E_1} \cong \text{Spin}(9)$ , we get

$$\mathbb{C}P_2 \cong F_4/F_4^\sigma = F_4/(F_4)_{E_1} \cong F_4/\text{Spin}(9).$$

We put  $\sigma_1 = \sigma$ , and we define the  $\mathbf{R}$ -linear map  $\sigma_2 : \mathfrak{J} \rightarrow \mathfrak{J}$  and  $\sigma_3 : \mathfrak{J} \rightarrow \mathfrak{J}$  respectively, as

$$\sigma_2 \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & -x_3 & \overline{x_2} \\ -\overline{x_3} & \xi_2 & -x_1 \\ x_2 & -\overline{x_1} & \xi_3 \end{pmatrix},$$

$$\sigma_3 \begin{pmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{pmatrix} = \begin{pmatrix} \xi_1 & x_3 & -\overline{x_2} \\ \overline{x_3} & \xi_2 & -x_1 \\ -x_2 & -\overline{x_1} & \xi_3 \end{pmatrix}.$$

For  $X = X(\xi, x)$ , we get

$$E_2 \circ X = \frac{1}{2} \begin{pmatrix} 0 & x_3 & 0 \\ \overline{x_3} & 2\xi_2 & x_1 \\ 0 & \overline{x_1} & 0 \end{pmatrix},$$

$$E_3 \circ X = \frac{1}{2} \begin{pmatrix} 0 & 0 & \overline{x_2} \\ 0 & 0 & x_1 \\ x_2 & \overline{x_1} & 2\xi_3 \end{pmatrix}.$$



Hence, for eigendecomposition

$$\mathfrak{J} = \mathfrak{J}_{\sigma_j} \oplus \mathfrak{J}_{-\sigma_j},$$

the following holds:

$$\begin{aligned} \mathfrak{J}_{\sigma_j} &= \{X \in \mathfrak{J} \mid \sigma_j X = X\}, \\ &= \{X \in \mathfrak{J} \mid E_j \circ X = \xi E_j, \xi \in \mathbf{R}\}, \\ \mathfrak{J}_{-\sigma_j} &= \{X \in \mathfrak{J} \mid \sigma_j X = -X\} \\ &= \{X \in \mathfrak{J} \mid E_j \circ X = \frac{1}{2}X\}, \end{aligned}$$

where any element  $X \in \mathfrak{J}_{\sigma_2}$ ,  $Y \in \mathfrak{J}_{-\sigma_2}$ ,  $Z \in \mathfrak{J}_{\sigma_3}$ ,  $W \in \mathfrak{J}_{-\sigma_3}$  are of the form

$$\begin{aligned} X &= \begin{pmatrix} \xi_1 & 0 & \bar{x}_2 \\ 0 & \xi_2 & 0 \\ x_2 & 0 & \xi_3 \end{pmatrix}, & Y &= \begin{pmatrix} 0 & x_3 & 0 \\ \bar{x}_3 & 0 & x_1 \\ 0 & \bar{x}_1 & 0 \end{pmatrix}, \\ Z &= \begin{pmatrix} \xi_1 & x_3 & 0 \\ \bar{x}_3 & \xi_2 & 0 \\ 0 & 0 & \xi_3 \end{pmatrix}, & W &= \begin{pmatrix} 0 & 0 & \bar{x}_2 \\ 0 & 0 & x_1 \\ x_2 & \bar{x}_1 & 0 \end{pmatrix}. \end{aligned}$$

At this time as in the case of  $\sigma_1$ , we get the following.

$$\begin{aligned} F_4^{\sigma_2} &= (F_4)_{E_2} \cong \text{Spin}(9), \\ F_4^{\sigma_3} &= (F_4)_{E_3} \cong \text{Spin}(9). \end{aligned}$$

**Proposition 3.6**  $\sigma_1, \sigma_2, \sigma_3$  are commutative cyclic involutions.

(Proof.) These are obtained by direct calculations.  $\square$

By using the symbols

$$\begin{aligned} F_4^{\sigma_i, \sigma_j} &= F_4^{\sigma_i} \cap F_4^{\sigma_j}, \quad (i \neq j), \\ F_4^{\sigma_1, \sigma_2, \sigma_3} &= F_4^{\sigma_1} \cap F_4^{\sigma_2} \cap F_4^{\sigma_3}, \end{aligned}$$

We get the following diagram.

$$\begin{array}{ccccc} & & F_4 & & \\ & \nearrow & \uparrow & \nwarrow & \\ F_4^{\sigma_1} & & F_4^{\sigma_2} & & F_4^{\sigma_3} \\ & \nwarrow & \uparrow & \nearrow & \\ & & F_4^{\sigma_1, \sigma_2, \sigma_3} & & \end{array}$$

In this diagram, we have

$$F_4^{\sigma_1, \sigma_2, \sigma_3} = F_4^{\sigma_1, \sigma_2} = F_4^{\sigma_2, \sigma_3} = F_4^{\sigma_3, \sigma_1} \cong \text{Spin}(8).$$

Hence this diagram means

$$\begin{array}{ccccc} & & F_4 & & \\ & \nearrow & \uparrow & \nwarrow & \\ \text{Spin}(9) & & \text{Spin}(9) & & \text{Spin}(9) \\ & \nwarrow & \uparrow & \nearrow & \\ & & \text{Spin}(8) & & \end{array}$$

The intersection of two different Spin(9) in this diagram is Spin(8).

By Theorem 3.5, we get the following corollary.

**Corollary 3.7** The Cayley plane  $\mathbb{C}P_2$  can be expressed as:

$$\begin{aligned} \mathbb{C}P_2 &\cong F_4/F_4^{\sigma_1} = F_4/(F_4)_{E_1} \cong F_4/\text{Spin}(9), \\ \mathbb{C}P_2 &\cong F_4/F_4^{\sigma_2} = F_4/(F_4)_{E_2} \cong F_4/\text{Spin}(9), \\ \mathbb{C}P_2 &\cong F_4/F_4^{\sigma_3} = F_4/(F_4)_{E_3} \cong F_4/\text{Spin}(9). \end{aligned}$$

### 3.6 Complex exceptional Lie group $F_4^C$

**Definition 3.8** We define the group  $F_4^C$  as the automorphism group of the complex Jordan algebra  $\mathfrak{J}^C$ :

$$F_4^C = \{\alpha \in \text{Iso}_{\mathbf{C}}(\mathfrak{J}^C) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)\}.$$

**Theorem 3.9** We can also define  $F_4^C$  as follows:

$$\begin{aligned} F_4^C &= \{\alpha \in \text{Iso}_{\mathbf{C}}(\mathfrak{J}^C) \mid \alpha(X \circ Y) = \alpha(X) \circ \alpha(Y)\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{C}}(\mathfrak{J}^C) \mid \alpha(X \times Y) = \alpha(X) \times \alpha(Y)\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{C}}(\mathfrak{J}^C) \mid \text{tr}(\alpha X, \alpha Y, \alpha Z) = \text{tr}(X, Y, Z) \\ &\quad (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{C}}(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, \\ &\quad (\alpha X, \alpha Y) = (X, Y)\} \\ &= \{\alpha \in \text{Iso}_{\mathbf{C}}(\mathfrak{J}^C) \mid \det(\alpha X) = \det X, \alpha E = E\}. \end{aligned}$$

(Proof.) See [16, Proposition 2.1.3.].  $\square$

For  $\alpha \in F_4$ , we define  $\alpha^C: \mathfrak{J}^C \rightarrow \mathfrak{J}^C$  as

$$\alpha^C(X_1 + iX_2) = \alpha(X_1) + i\alpha(X_2).$$

By identifying  $\alpha$  and  $\alpha^C$ , we can consider  $F_4$  as the subgroup of  $F_4^C$ :

$$F_4 \subseteq F_4^C.$$

When we extend a map  $\alpha \in \text{Hom}_{\mathbf{R}}(\mathbb{C}, \mathbb{C})$  to the map of  $\text{Hom}_{\mathbf{C}}(\mathfrak{J}^C, \mathfrak{J}^C)$ , the following diagram is commutative.

$$\begin{array}{ccc} \text{Hom}_{\mathbf{R}}(\mathbb{C}, \mathbb{C}) & \rightarrow & \text{Hom}_{\mathbf{C}}(\mathbb{C}^C, \mathbb{C}^C) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathbf{R}}(\mathfrak{J}, \mathfrak{J}) & \rightarrow & \text{Hom}_{\mathbf{C}}(\mathfrak{J}^C, \mathfrak{J}^C) \end{array}$$

That is,  $(\tilde{\alpha})^C = \tilde{\alpha}^C$ . Then we get the following diagram.

$$\begin{array}{ccc} F_4 & \rightarrow & F_4^C \\ \uparrow & & \uparrow \\ G_2 & \rightarrow & G_2^C \end{array}$$

We consider the subgroup  $(F_4^C)^{\tau}$ :

$$(F_4^C)^{\tau} = \{\alpha \in F_4^C \mid \alpha \tau = \tau \alpha\}.$$

Then  $(F_4^C)^{\tau}$  is isomorphic to  $F_4$ :

$$F_4 = (F_4^C)^\tau \subseteq F_4^C.$$

**Proposition 3.10**  $\tau, \gamma, \sigma$  are commutative, as elements of  $F_4^C$ , respectively:

$$\tau\gamma = \gamma\tau, \quad \tau\sigma = \sigma\tau, \quad \gamma\sigma = \sigma\gamma.$$

(Proof.) These are obtained by direct calculations.  $\square$

Then,  $F_{4(4)}$  and  $F_{4(-20)}$  are represented as an invariant group by  $\tau\gamma$  and  $\tau\sigma$ , respectively:

$$F_{4(4)} = (F_4^C)^{\tau\gamma} \subseteq F_4^C,$$

$$F_{4(-20)} = (F_4^C)^{\tau\sigma} \subseteq F_4^C.$$

For  $\alpha \in F_4$  and  $X = X(\xi, x) \in \mathfrak{J}$ , from  $\alpha(-x_j) = -\alpha(x_j)$  ( $j = 2, 3$ ), we get

$$\sigma\alpha X(\xi, x) = \sigma X(\xi, \alpha x) = \alpha\sigma X(\xi, x).$$

Hence we have  $G_2 \subseteq F_{4(-20)}$ .

$G_{2(2)} \subseteq F_{4(4)}$  is obvious.

**Theorem 3.11** The following inclusive relations hold:

$$G_2 \subseteq F_4, \quad G_2 \subseteq F_{4(-20)}, \quad G_{2(2)} \subseteq F_{4(4)}.$$

We get the following diagram.

$$\begin{array}{ccccc} & & F_4^C & & \\ & \nearrow & \uparrow & \nwarrow & \\ F_{4(-20)} & & F_4 & & F_{4(4)} \\ & \nwarrow & \uparrow & \nearrow & \\ & & G_2 & & G_{2(2)} \end{array}$$

#### 4. Construction of $(F_4^C)^\gamma$ and $(F_4^C)^\sigma$

Yokota constructed  $(F_4^C)^\gamma$  and  $(F_4^C)^\sigma$ , concretely. Here we describe the ideas.

Let  $\mathfrak{J}_{\mathbf{H}}$  denote all  $3 \times 3$  Hermitian matrices with entries in  $\mathbf{H}$ .

$$\mathfrak{J}_{\mathbf{H}} = \{X \in M(3, \mathbf{H}) \mid X^* = X\}$$

Any element  $M \in \mathfrak{J}_{\mathbf{H}}$  is of the form

$$M = \begin{pmatrix} \xi_1 & m_3 & \overline{m_2} \\ \overline{m_3} & \xi_2 & m_1 \\ m_2 & \overline{m_1} & \xi_3 \end{pmatrix},$$

where  $\xi_i \in \mathbf{R}$ ,  $m_j \in \mathbf{H}$ . And, for  $\mathbf{a} = (a_1, a_2, a_3) \in \mathbf{H}^3$ , we take

$$F(\mathbf{a}e_4) = \begin{pmatrix} 0 & a_3e_4 & -a_2e_4 \\ -a_3e_4 & 0 & a_1e_4 \\ a_2e_4 & -a_1e_4 & 0 \end{pmatrix}.$$

By identifying  $M + \mathbf{a} \in \mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3$  and  $M + F(\mathbf{a}e_4) \in \mathfrak{J}$ , we get the following as vector spaces:

$$\mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3 = \mathfrak{J}.$$

In  $\mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3$ , we define a Freudenthal multiplication  $X \times Y$  and an inner product  $(X, Y)$  as follows:

$$(M + \mathbf{a}) \times (N + \mathbf{b}) = (M \times N - \frac{1}{2}(\mathbf{a}^* \mathbf{b} + \mathbf{b}^* \mathbf{a})) - \frac{1}{2}(\mathbf{a}N + \mathbf{b}M),$$

$$(M + \mathbf{a}, N + \mathbf{b}) = (M, N) + 2(\mathbf{a}, \mathbf{b}).$$

These make  $\mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3$  and  $\mathfrak{J}$  isomorphic, which keeps the inner product as algebras. Then we have

$$\gamma(M + \mathbf{a}) = M - \mathbf{a}.$$

By considering the complexification  $(\mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3)^C = \mathfrak{J}_{\mathbf{H}^C} \oplus (\mathbf{H}^C)^3$ , we have

$$\mathfrak{J}_{\mathbf{H}^C} \oplus (\mathbf{H}^C)^3 = \mathfrak{J}^C.$$

We define  $Sp(n, K)$  as

$$Sp(n, K) = \{A \in M(n, K) \mid A^*A = E\}, \quad K = \mathbf{H}, \mathbf{H}^C.$$

As a notation, we write  $Sp(n) = Sp(n, \mathbf{H})$ .

**Theorem 4.1** (1)  $F_4^\gamma \cong (Sp(1) \times Sp(3))/\mathbf{Z}_2$ ,

where  $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$ .

(2)  $(F_4^C)^\gamma \cong (Sp(1, \mathbf{H}^C) \times Sp(3, \mathbf{H}^C))/\mathbf{Z}_2$ ,

where  $\mathbf{Z}_2 = \{(1, E), (-1, -E)\}$ .

(Proof.) (1) We define  $\varphi : Sp(1, \mathbf{H}) \times Sp(3, \mathbf{H}) \rightarrow F_4^\gamma$  by  $\varphi(p, A)(M + \mathbf{a}) = AMA^* + p\mathbf{a}A^*$ ,  $M + \mathbf{a} \in \mathfrak{J}_{\mathbf{H}} \oplus \mathbf{H}^3 = \mathfrak{J}$ .

Then  $\varphi$  is a homomorphism and onto with  $\text{Ker}\varphi = \{(1, E), (-1, -E)\} = \mathbf{Z}_2$ .

(2) Similarly, we define  $\varphi : Sp(1, \mathbf{H}^C) \times Sp(3, \mathbf{H}^C) \rightarrow (F_4^C)^\gamma$  by  $\varphi(p, A)(M + \mathbf{a}) = AMA^* + p\mathbf{a}A^*$ ,  $M + \mathbf{a} \in \mathfrak{J}_{\mathbf{H}^C} \oplus (\mathbf{H}^C)^3 = \mathfrak{J}^C$ .  $\square$

On the other hand, for  $F_4^\sigma$  and  $(F_4^C)^\sigma$ , by showing that these are the universal covering group of  $\text{SO}(9)$  and  $\text{SO}(9, \mathbf{C})$  respectively, we obtain the following theorem.

**Theorem 4.2** (1)  $F_4^\sigma \cong \text{Spin}(9)$ .

(2)  $(F_4^C)^\sigma \cong \text{Spin}(9, \mathbf{C})$ .

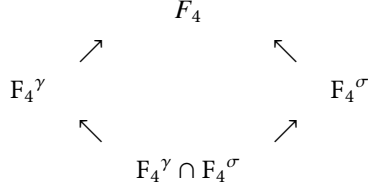
#### 5. Arrangement of subgroups of $F_4^C$

Let  $L$  be a subgroup of  $F_4^C$  and let  $\theta \in F_4^C$ , then we define  $L^\theta$  as the following way :

$$L^\theta = \{ \alpha \in L \mid \alpha\theta = \theta\alpha \}.$$

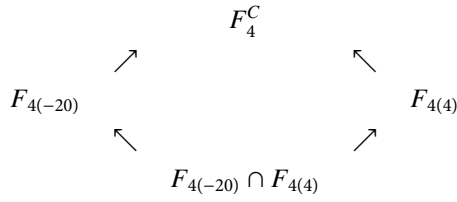
In this notation, we get  $G_2^\sigma = G_2$ .

By using  $\gamma$  and  $\sigma$ , we can also make the following diagram for  $F_4$ .

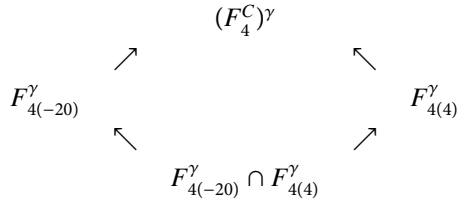


In this diagram,  $F_4/F_4^\sigma \cong \mathbb{C}P_2$  holds.

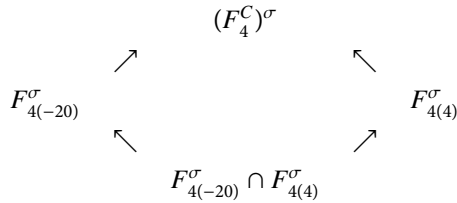
The arrangement of the non-compact groups  $F_{4(-20)}$  and  $F_{4(4)}$  in  $F_4^C$  is as shown in the following diagram.



By using  $\gamma$  for this diagram, we can get the following diagram of the  $\gamma$  sequence.



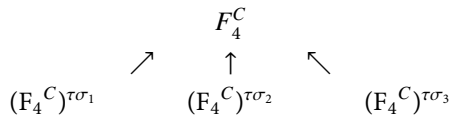
Similarly, we can get the following diagram of the  $\sigma$  sequence.



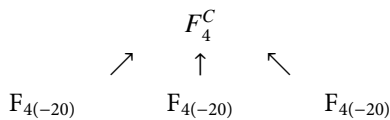
As in the case of  $\sigma = \sigma_1$ , we have

$$\begin{aligned}
 (F_4^C)^{\tau\sigma_2} &\cong F_{4(-20)}, \\
 (F_4^C)^{\tau\sigma_3} &\cong F_{4(-20)}.
 \end{aligned}$$

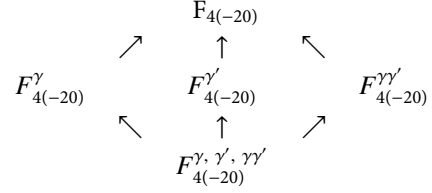
Hence, we get the following diagram.



This diagram means



In each  $F_{4(-20)} = (F_4^C)^{\tau\sigma_j}$ , the following diagram can be obtained as a hierarchical structure by using commutative cyclic involutions  $\gamma, \gamma', \gamma\gamma' \in (F_4^C)^{\tau\sigma_j}$ .



The intersection of two different  $F_{4(-20)}^\gamma, F_{4(-20)}^{\gamma'}, F_{4(-20)}^{\gamma\gamma'}$  are equal:

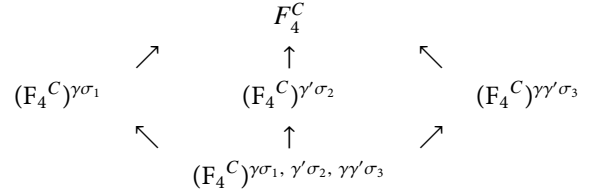
$$\begin{aligned}
 F_{4(-20)}^\gamma \cap F_{4(-20)}^{\gamma'} &= F_{4(-20)}^\gamma \cap F_{4(-20)}^{\gamma\gamma'} = F_{4(-20)}^{\gamma'} \cap F_{4(-20)}^{\gamma\gamma'} \\
 &= F_{4(-20)}^{\gamma, \gamma', \gamma\gamma'}.
 \end{aligned}$$

By combining commutative cyclic involutions  $\sigma_1, \sigma_2, \sigma_3$  and  $\gamma, \gamma', \gamma\gamma'$ , we get the following proposition.

**Proposition 5.1**  $\gamma\sigma_1, \gamma'\sigma_2, \gamma\gamma'\sigma_3$  are commutative cyclic involutions.

(Proof.) These are obtained by direct calculations.  $\square$

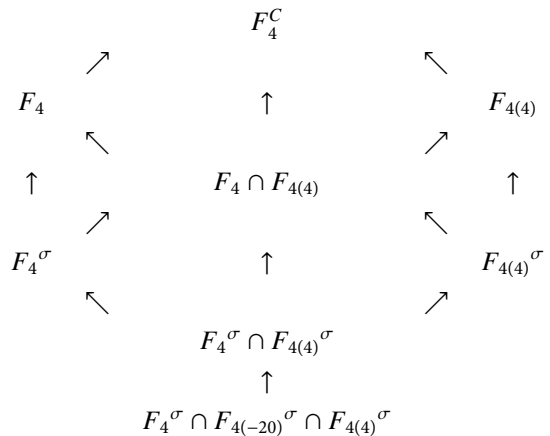
Hence, we get the following diagram.



The intersection of two different  $(F_4^C)^{\gamma\sigma_1}, (F_4^C)^{\gamma'\sigma_2}, (F_4^C)^{\gamma\gamma'\sigma_3}$  are equal:

$$\begin{aligned}
 (F_4^C)^{\gamma\sigma_1} \cap (F_4^C)^{\gamma'\sigma_2} &= (F_4^C)^{\gamma\sigma_1} \cap (F_4^C)^{\gamma\gamma'\sigma_3} \\
 &= (F_4^C)^{\gamma'\sigma_2} \cap (F_4^C)^{\gamma\gamma'\sigma_3} = (F_4^C)^{\gamma\sigma_1, \gamma'\sigma_2, \gamma\gamma'\sigma_3}.
 \end{aligned}$$

Moreover, we can get larger diagrams by combining  $F_4$ -type Lie groups and involutions. For example, we can get the following diagram.



We can make various other diagrams by changing groups and involutions.

**Problem** Investigate the structure of these diagrams as Lie groups in detail.

In these diagrams using Lie groups and involutions, there may be interesting facts that we do not yet know in the concrete construction and application. The Yokota-style method has the potential to investigate the structure of these exceptional Lie groups from the perspective of Lie groups (rather than Lie algebras).

## 6. Conclusion and future direction

### 6.1 Realization of concrete subgroups

By using the theory of Lie algebras, we can abstractly understand the subgroups of the Lie group. However, using the Yokota-style construction method opens up the possibility of concrete realization as groups.

In this paper, we dealt with the compact group  $F_4$ , the non-compact groups  $F_{4(4)}$ ,  $F_{4(-20)}$  and the complex compact group  $F_4^C$  and we investigated the arrangement of their subgroups using involutions. These studies are expected to be further refined. Moreover, it is conceivable to extend the research to the exceptional Lie groups of type  $E$ .

### 6.2 Relationship with M-theory in physics

In recent years, M-theory has been energetically studied in mathematical physics. And M-theory is deeply linked to the exceptional Lie groups.

In M-theory, we consider the fiber bundle  $\pi : M \rightarrow Y$  with the fiber  $\mathbb{C}P_2$  and the structure group  $F_4$ . At this time, the base space  $Y$  is an 11-dimensional manifold, and the total space  $M$  is a 27-dimensional manifold. In this  $F_4 - \mathbb{C}P_2$  bundle  $(M, \pi, Y, F_4; \mathbb{C}P_2)$ , H. Sati (2009) investigated the question of whether when the 11-dimensional base manifold  $Y$  has a Spin, String, or Five-brane structure, it leads to a similar structure in the 27-dimensional manifold  $M$ . From a mathematical physics point of view, it is necessary to study the further connection between M-theory and the exceptional Lie groups.

From a mathematical point of view, we can also consider the fiber bundle with the complex Cayley plane  $\mathbb{C}^C P_2$  as the fiber and the exceptional Lie group  $E_6$  as the structure group. Namely, a fiber bundle  $\mu : N \rightarrow Z$  with the fiber  $\mathbb{C}^C P_2$  and the structure group  $E_6$  can be mathe-

atically considered:

$$(N, \mu, Z, E_6, ; \mathbb{C}^C P_2).$$

$\mathbb{C}^C P_2$  is no longer projective geometry, but it may have a mathematically rich structure.

The question of mathematically investigating the relationship between the  $E_6 - \mathbb{C}^C P_2$  bundle and the  $F_4 - \mathbb{C}P_2$  bundle can be considered. In this way, in relation to M-theory, our future expectation is to investigate fiber bundles where the fiber is the Cayley plane and the structure group is the exceptional Lie group in the future.

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# 例外型リー群 $F_4^C$ における部分群の 横田流の手法を用いた具体的な配置について

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(受領日：2021年5月31日)

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要約：本研究では、ジョルダン代数、split ジョルダン代数、及びその複素化を用いて、例外型 Lie 群  $F_4^C$  の部分群を横田流の手法で具体的に構成する。横田流の手法には、非コンパクト群や複素 Lie 群を自然に構成できるという特徴がある。本論文では最初に、 $G_2$  型のリー群を調べるためにケーリー代数を定義し、次にジョルダン代数を用いて  $F_4$  型の Lie 群と対合へ拡張する。最後に、 $F_4$  型の Lie 群の部分群の配置を対合を用いて考察する。特に、 $F_4$  型の 2 つの非コンパクト群とその部分群の配置を、その対合不変部分群として考察する。