高知工科大学

基礎数学ワークブック

(1999年度版)

番外編 *No.*2

「 ベクトル解析入門 」

解答

＜ 2 ページ. 平面上の道のり 2 ＞

問 1 の解答

$$\ell = \int_0^\theta \sqrt{\left(-r\sin t\right)^2 + \left(r\cos t\right)^2}dt = \int_0^\theta r dt = \theta r$$

問 2 の解答

$$\ell = \int_a^b \sqrt{1 + \left(f'(t)\right)^2}dt$$

＜ 3ページ. 線と面 ＞

問 1 の解答

$$S = \int_0^3 (-x^2 + 3x)dx = \left[-\frac{x^3}{3} + \frac{3}{2}x^2\right]_0^3$$
$$= -9 + \frac{27}{2}$$
$$= \frac{9}{2}$$

問 2 の解答

$$S = \pi R^2 \quad , \quad \ell(r) = 2\pi r$$

問 3 の解答

$$S = \int_0^R \theta r dr = \frac{\theta}{2}R^2 \quad , \quad \ell(r) = \theta r$$

＜ 4ページ. 面と立体 ＞

問の解答

$$
\begin{aligned}
V &= \int_{-r}^{r} \pi\left(\sqrt{r^2 - x^2}\right)^2 dx = \int_{-r}^{r} \pi\left(r^2 - x^2\right) dx \\
&= \left[\pi r^2 x - \frac{\pi}{3}x^3\right]_{-r}^{r} \\
&= \pi r^3 - \frac{\pi}{3}r^3 - \left(-\pi r^3 + \frac{\pi}{3}r^3\right) \\
&= \frac{2}{3}\pi r^3 - \left(-\frac{2}{3}\pi r^3\right) \\
&= \frac{4}{3}\pi r^3
\end{aligned}
$$

< 5ページ. 質量と重心 1 >

問の解答

$$g = \frac{1}{M}\left\{m_1x_1 + m_2x_2 + \cdots + m_nx_n\right\}$$

＜7ページ. 質量と重心 3＞

問の解答

$$
\begin{aligned}
M &= \int_0^2 f(x)dx = \int_0^2 (-x^2 + 2x)dx \\
&= \left[-\frac{x^3}{3} + x^2\right]_0^2 \\
&= -\frac{8}{3} + 4 \\
&= \frac{4}{3}
\end{aligned}
$$

$$
\begin{aligned}
g &= \frac{1}{M}\int_0^2 xf(x)dx = \frac{1}{\frac{4}{3}}\int_0^2 (-x^3 + 2x^2)dx \\
&= \frac{3}{4}\left[-\frac{x^4}{4} + \frac{2}{3}x^3\right]_0^2 \\
&= \frac{3}{4}\left\{-4 + \frac{16}{3}\right\} \\
&= \frac{3}{4} \times \frac{4}{3} \\
&= 1
\end{aligned}
$$

＜ 8ページ. 質量と重心 4 ＞

問の解答

$$g_Y = \frac{1}{M}\iint_D yf(x,y)dxdy = \frac{1}{3}\iint_{D_1} ydxdy + \frac{1}{3}\iint_{D_2} ydxdy$$

$$= \frac{1}{3}\int_0^1 \Big\{\int_0^{2x} ydy\Big\}dx + \frac{1}{3}\int_1^3 \Big\{\int_0^{-x+3} ydy\Big\}dx$$

$$= \frac{1}{3}\int_0^1 \Big\{\Big[\frac{y^2}{2}\Big]_{y=0}^{y=2x}\Big\}dx + \frac{1}{3}\int_1^3 \Big\{\Big[\frac{y^2}{2}\Big]_{y=0}^{y=-x+3}\Big\}dx$$

$$= \frac{1}{3}\int_0^1 2x^2dx + \frac{1}{3}\int_1^3 \frac{(-x+3)^2}{2}dx$$

$$= \frac{1}{3}\Big[\frac{2}{3}x^3\Big]_0^1 + \frac{1}{3}\Big[-\frac{(-x+3)^3}{6}\Big]_1^3$$

$$= \frac{2}{9} + \frac{1}{3}\Big(0 + \frac{8}{6}\Big) = \frac{2}{9} + \frac{4}{9}$$

$$= \frac{2}{3}$$

< 9ページ. 回転体の表面積 1 >

問の解答

$S = \pi m\sqrt{1+m^2}\, b^2 - \pi m\sqrt{1+m^2}\, a^2$

＜ 10 ページ. 回転体の表面積 2 ＞

問の解答

$$S = \int_1^3 6\pi dx = 12\pi$$

< 11 ページ. 回転体の表面積 3 >

問の解答

$$S = \int_{-R}^{R} 2\pi y\sqrt{1+(y')^2}dx = \int_{-R}^{R} 2\pi R dx = 4\pi R^2$$

< 13 ページ. 平面上の運動 2 >

問 1 の解答

$$\begin{aligned}\frac{d^2\boldsymbol{r}}{dt^2} = \frac{d\boldsymbol{v}}{dt} &= \left(\frac{d}{dt}\big(-6\pi\sin(2\pi t)\big)\ ,\ \frac{d}{dt}\big(6\pi\cos(2\pi t)\big)\right) \\ &= \big(-12\pi^2\cos(2\pi t), -12\pi^2\sin(2\pi t)\big)\end{aligned}$$

$$\left|\frac{d^2\boldsymbol{r}}{dt^2}\right| = \sqrt{\big(-12\pi^2\cos(2\pi t)\big)^2 + \big(-12\pi^2\sin(2\pi t)\big)^2} = 12\pi^2$$

問 2 の解答

$$S = \int_0^{\frac{1}{4}}\left|\frac{d\boldsymbol{r}}{dt}\right|dt = \int_0^{\frac{1}{4}} 6\pi dt = 6\pi \times \frac{1}{4} = \frac{3}{2}\pi$$

＜ 14 ページ. 平面上の線積分 1 ＞

問 1 の解答

$$\int_C xydt = \int_{-1}^{1} x(t)y(t)dt = \int_{-1}^{1} t \times t^2 dt = \left[\frac{1}{4}t^4\right]_{-1}^{1} = 0$$

問 2 の解答

$$\begin{aligned}\int_C (x+y)dt &= \int_0^{\frac{\pi}{2}} (3\cos t + 3\sin t)dt = \Big[3\sin t - 3\cos t\Big]_0^{\frac{\pi}{2}} \\ &= 3\sin\left(\frac{\pi}{2}\right) - 3\cos\left(\frac{\pi}{2}\right) - 3\sin 0 + 3\cos 0 \\ &= 3 - 0 - 0 + 3 \\ &= 6\end{aligned}$$

< 16 ページ. 平面上の線積分 3 >

問の解答

(1) $\displaystyle\int_C (x^2+y^2)ds$

$$= \int_0^{2\pi} (r^2\cos^2 t + r^2\sin^2 t)\sqrt{(-r\sin t)^2+(r\cos t)^2}dt$$

$$= \int_0^{2\pi} r^2 \times r dt$$

$$= 2\pi r^3$$

(2) $\displaystyle\int_C (x+y)ds$

$$= \int_0^{2\pi} (r\cos t + r\sin t)\sqrt{(-r\sin t)^2+(r\cos t)^2}dt$$

$$= \int_0^{2\pi} (r\cos t + r\sin t) r dt$$

$$= r^2\Big[\sin t - \cos t\Big]_0^{2\pi}$$

$$= r^2\Big\{\sin(2\pi) - \cos(2\pi) - \sin 0 + \cos 0\Big\}$$

$$= 0$$

< 17ページ. 平面上の線積分 4 >

問の解答

(1) $\displaystyle\int_C (x+y)dx = \int_0^1 \big(x(t)+y(t)\big)\frac{dx}{dt}dt$

$\displaystyle= \int_0^1 (t+\sqrt{t})dt$

$\displaystyle= \left[\frac{t^2}{2}+\frac{2}{3}t^{\frac{3}{2}}\right]_0^1$

$\displaystyle= \frac{1}{2}+\frac{2}{3}$

$\displaystyle= \frac{7}{6}$

(2) $\displaystyle\int_C (x+y)dy = \int_0^1 \big(x(t)+y(t)\big)\frac{dy}{dt}dt$

$\displaystyle= \int_0^1 (t+\sqrt{t})\frac{1}{2\sqrt{t}}dt = \int_0^1 \left(\frac{1}{2}\sqrt{t}+\frac{1}{2}\right)dt$

$\displaystyle= \left[\frac{1}{3}t^{\frac{3}{2}}+\frac{1}{2}t\right]_0^1$

$\displaystyle= \frac{1}{3}+\frac{1}{2}$

$\displaystyle= \frac{5}{6}$

< 18 ページ. 平面上の線積分 5 >

問の解答

$$S_2 = \int_a^b \psi(x)dx = -\int_b^a ydx = -\int_C ydx$$

< 19 ページ. 平面上の線積分 6 >

問の解答

$$S = \int_{C_2} xdy - \int_{-C_1} xdy = \int_{C_2} xdy + \int_{C_1} xdy = \int_C xdy$$

< 20 ページ. グリーンの定理 1 >

問の解答

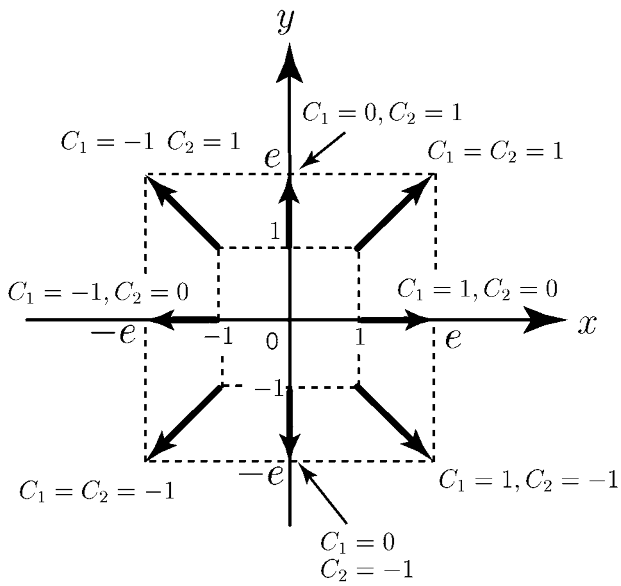
$$\iint_D \frac{\partial f}{\partial y} dxdy = -\int_b^a f\bigl(x, \psi(x)\bigr)dx - \int_a^b f\bigl(x, \varphi(x)\bigr)dx = -\int_C f(x, y)dx$$

< 22 ページ. 平面上の流れ 1 >

問の解答

$$\frac{dx}{dt} = x \ \Rightarrow x(t) = C_1 e^t$$

$$\frac{dy}{dt} = y \ \Rightarrow y(t) = C_2 e^t$$



< 23ページ. 平面上の流れ 2 >

問の解答

$$\mathrm{div}(\boldsymbol{v}) = \mathrm{div}(-x, y) = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(y) = -1 + 1 = 0$$

< 24 ページ. 平面上の流れ 3 >

問の解答

(1) $\boldsymbol{v} = (2x, 2y)$

$\mathrm{div}(\boldsymbol{v}) = 2 + 2 = 4$

$\mathrm{rot}(\boldsymbol{v}) = 0$

(2) $\boldsymbol{v} = (2x - y, 2y + x)$

$\mathrm{div}(\boldsymbol{v}) = 2 + 2 = 4$

$\mathrm{rot}(\boldsymbol{v}) = 1 - (-1) = 2$

(3) $\boldsymbol{v} = (2x + 3y, 4x - 5y)$

$\mathrm{div}(\boldsymbol{v}) = 2 - 5 = -3$

$\mathrm{rot}(\boldsymbol{v}) = 4 - 3 = 1$

< 25 ページ. 平面上の流れ 4 >

問の解答

$U(x,y) = -\dfrac{1}{2}x^2 - \dfrac{1}{2}y^2$ とおくと

$$\left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}\right) = \Big(-(-x), -(-y)\Big) = (x,y) = \boldsymbol{v}$$

より　$U(x,y) = -\dfrac{1}{2}x^2 - \dfrac{1}{2}y^2$ がポテンシャルである。

< 28 ページ. 平面のベクトル場の線積分 3 >

問の解答

$$\iint_D \mathrm{rot}(\boldsymbol{F})dxdy = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)dxdy$$

$$= \int_C (f_1 dx + f_2 dy) = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}$$

< 29 ページ. 空間のベクトル 1 >

問の解答

(1) $|\boldsymbol{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

(2) $|\boldsymbol{b}| = \sqrt{3^2 + 0^2 + (-1)^2} = \sqrt{10}$

(3) $\boldsymbol{a} \cdot \boldsymbol{b} = 1 \times 3 + 2 \times 0 + 3 \times (-1) = 0$

(4) $\theta = 90^\circ = \dfrac{\pi}{2}$

(5) $$\begin{aligned} S &= |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \sin\theta \\ &= \sqrt{14} \times \sqrt{10} \times \sin\left(\frac{\pi}{2}\right) \\ &= \sqrt{140} \\ &= 2\sqrt{35} \end{aligned}$$

< 30 ページ. 空間のベクトル 2 >

問の解答

(1) $\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & 2 & 0 \\ 1 & 4 & 0 \end{vmatrix} = 10\boldsymbol{k} = (0,\ 0,\ 10)$

(2) $\boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 4 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 12\boldsymbol{i} - 3\boldsymbol{j} - 3\boldsymbol{k} = (12,\ -3,\ -3)$

(3) $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (0,\ 0,\ 10) \cdot (1,\ 1,\ 3) = 30$

(4) $(\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (12,\ -3,\ -3) \cdot (3,\ 2,\ 0) = 30$

< 31 ページ. 空間の運動 >

問の解答

$\boldsymbol{v} = (-\sin t\ ,\ \cos t\ ,\ 0.2)$

$$|\boldsymbol{v}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (0.2)^2}$$

$$= \sqrt{1 + 0.04}$$

$$= \sqrt{1.04}$$

$$s = \int_0^{2\pi} |\boldsymbol{v}| dt = \int_0^{2\pi} \sqrt{1.04} dt = 2\pi\sqrt{1.04}$$

< 32 ページ. 空間の線積分 >

問の解答

(1)
$$\begin{aligned}\int_C f\,ds &= \int_C \big(x(t)+y(t)+z(t)\big)\left|\frac{dr}{dt}\right|dt \\ &= \int_0^{2\pi}(\cos t+\sin t+0.2t+1)\sqrt{1.04}dt \\ &= \sqrt{1.04}\times\big[\sin t-\cos t+0.1t^2+t\big]_0^{2\pi} \\ &= (0.4\pi^2+2\pi)\sqrt{1.04}\end{aligned}$$

(2)
$$\begin{aligned}\int_C f\,dz &= \int_0^{2\pi}\big(x(t)+y(t)+z(t)\big)\frac{dz}{dt}dt \\ &= \int_0^{2\pi}(\cos t+\sin t+0.2t+1)0.2\,dt \\ &= 0.2\times\big[\sin t-\cos t+0.1t^2+t\big]_0^{2\pi} \\ &= 0.2(0.4\pi^2+2\pi) \\ &= 0.08\pi^2+0.4\pi\end{aligned}$$

＜ 33 ページ. 空間のベクトル場の線積分 ＞

問の解答

$$
\begin{aligned}
\int_C \boldsymbol{F}\cdot d\boldsymbol{r} &= \int_0^{2\pi}\left\{\cos^2 t+\sin^2 t\right\}(-\sin t)dt \\
&\quad + \int_0^{2\pi}\frac{\sin t}{\cos t}\times\cos t\,dt \\
&\quad + \int_0^{2\pi}(0.2t+1)\times 0.2\,dt \\
&= \left[\cos t\right]_0^{2\pi} + \left[-\cos t\right]_0^{2\pi} + 0.2\left[0.1t^2+t\right]_0^{2\pi} \\
&= 0.2(0.4\pi^2+2\pi) \\
&= 0.08\pi^2+0.4\pi
\end{aligned}
$$

＜ 34 ページ. 平面のパラメーター表示 ＞

問の解答

$$\frac{\partial \boldsymbol{r}}{\partial v} = \left(\frac{\partial x}{\partial v}\ ,\ \frac{\partial y}{\partial v}\ ,\ \frac{\partial z}{\partial v}\right) = (b_1\ ,\ b_2\ ,\ b_3) = \boldsymbol{b}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$$

$<$ 35 ページ. 球面のパラメーター表示 $>$

問の解答

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$$

$$= (-R\cos v \sin u \ , \ R\cos v \cos u \ , \ 0) \times (-R\sin v \cos u \ , \ -R\sin v \sin u \ , \ R\cos v)$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -R\cos v \sin u & R\cos v \cos u & 0 \\ -R\sin v \cos u & -R\sin v \sin u & R\cos v \end{vmatrix}$$

$$= (R^2\cos^2 v \cos u)\boldsymbol{i} + (R^2\cos^2 v \sin u)\boldsymbol{j} + (R^2\cos v \sin v \sin^2 u + R^2\cos v \sin v \cos^2 u)\boldsymbol{k}$$

$$= R\cos v(R\cos v \cos u \ , \ R\cos u \sin u \ , \ R\sin v)$$

$$= R\cos v \ \boldsymbol{r}(u, v)$$

＜ 37 ページ. 曲面の面積 2 ＞

問の解答

$$D = \left\{(u,v)\ :\ 0 \leqq u \leqq 2\pi\ ,\ -\frac{\pi}{2} \leqq v \leqq \frac{\pi}{2}\right\}$$

$$S = \iint_D \left|\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right| dudv = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{\int_0^{2\pi} R^2 \cos v du\right\} dv = 2\pi R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos v dv$$

$$= 2\pi R^2 \Big[\sin v\Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi R^2$$

< 38 ページ. スカラー場の面積分 >

問の解答

$$
\begin{aligned}
\int_S 1dS &= \int_0^{2\pi} \left\{ \int_a^b 1f(u)\sqrt{1+\big(f'(u)\big)^2}du \right\} dv \\
&= \int_a^b \left\{ \int_0^{2\pi} f(u)\sqrt{1+\big(f'(u)\big)^2}dv \right\} du \\
&= \int_a^b 2\pi f(u)\sqrt{1+\big(f'(u)\big)^2}du \\
&= \int_a^b 2\pi f(x)\sqrt{1+\big(f'(x)\big)^2}dx
\end{aligned}
$$

< 40 ページ. ベクトル場の表面積分 2 >

問の解答

$$\int_S \boldsymbol{F}\cdot d\boldsymbol{S} = \int_0^{2\pi}\left\{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\Big(\frac{1+\cos(2v)}{2}\big(a\cos u + b\sin u\big) + \frac{c}{2}\sin(2v)\Big)dv\right\}du$$

$$= \int_0^{2\pi}\left\{(a\cos u + b\sin u)\left[\frac{v}{2}+\frac{1}{4}\sin(2v)\right]_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}} + \frac{c}{2}\left[-\frac{1}{2}\cos(2v)\right]_{v=-\frac{\pi}{2}}^{v=\frac{\pi}{2}}\right\}du$$

$$= \frac{\pi}{2}\int_0^{2\pi}(a\cos u + b\sin u)du = \frac{\pi}{2}\Big[a\sin u - b\cos u\Big]_0^{2\pi} = 0$$

< 41 ページ. ベクトル場の表面積分 3 >

問の解答

$$\int_S (0,0,f_3)\cdot d\boldsymbol{S} = -\iint_D \begin{vmatrix} 0 & 0 & f_3 \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} dxdy = -\iint_D f_3\bigl(x,y,z(x,y)\bigr)dxdy$$

< 42 ページ. 体積積分 1 >

問の解答

$$\int_V \varphi dV = \int_0^3 \left\{ \int_0^2 \left\{ \int_0^1 \varphi(x,y,z)dz \right\} dy \right\} dx$$

< 43 ページ. 体積積分 2 >

問の解答

$$\begin{aligned}
\int_V \varphi dV &= \int_0^4 \left\{ \int_0^3 \left\{ \int_0^2 (x+y+z)dz \right\} dy \right\} dx \\
&= \int_0^4 \left\{ \int_0^3 \left[(x+y)z + \frac{z^2}{2} \right]_{z=0}^{z=2} dy \right\} dx = \int_0^4 \left\{ \int_0^3 (2x+2y+2)dy \right\} dx \\
&= \int_0^4 \left\{ \left[2xy + y^2 + 2y \right]_{y=0}^{y=3} \right\} dx = \int_0^4 (6x+15)dx \\
&= \left[3x^2 + 15x \right]_{x=0}^{x=4} \\
&= 108
\end{aligned}$$

< 45 ページ. 回転 1 >

問の解答

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{z} = (\omega_1, \omega_2, \omega_3) \times (x, y, z)$$

$$= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\boldsymbol{i} + (\omega_3 x - \omega_1 z)\boldsymbol{j} + (\omega_1 y - \omega_2 x)\boldsymbol{k}$$

より $v_1 = \omega_2 z - \omega_3 y$, $v_2 = \omega_3 x - \omega_1 z$, $v_3 = \omega_1 y - \omega_2 x$

＜ 46 ページ. 回転 2 ＞

問の解答

(1) $\boldsymbol{\omega} = (0, 0, \theta)$

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \theta \\ x & y & z \end{vmatrix} = (-y\theta\ ,\ x\theta\ ,\ 0)$$

$$\mathrm{rot}\boldsymbol{v} = \left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x\theta)\ ,\ \frac{\partial}{\partial z}(-y\theta) - \frac{\partial}{\partial x}(0)\ ,\ \frac{\partial}{\partial x}(x\theta) - \frac{\partial}{\partial y}(-y\theta) \right)$$

$$= (0, 0, 2\theta) = 2\boldsymbol{\omega}$$

(2) $\boldsymbol{\omega} = (\omega_1\ ,\ \omega_2\ ,\ \omega_3)$

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = (\omega_2 z - \omega_3 y\ ,\ \omega_3 x - \omega_1 z\ ,\ \omega_1 y - \omega_2 x)$$

$$\mathrm{rot}\boldsymbol{v} = \big(\omega_1 - (-\omega_1)\ ,\ \omega_2 - (-\omega_2)\ ,\ \omega_3 - (-\omega_3)\big)$$

$$= (2\omega_1\ ,\ 2\omega_2\ ,\ 2\omega_3) = 2\boldsymbol{\omega}$$

< 47 ページ. 発散 >

問の解答

(1) $\boldsymbol{v} = (ax - by\ ,\ ay + bx\ ,\ cz)$

$$\begin{aligned}\mathrm{div}\boldsymbol{v} &= \frac{\partial}{\partial x}(ax - by) + \frac{\partial}{\partial y}(ay + bx) + \frac{\partial}{\partial z}(cz)\\ &= a + b + c\end{aligned}$$

(2) $\boldsymbol{v} = (\omega_2 z - \omega_3 y\ ,\ \omega_3 x - \omega_1 z\ ,\ \omega_1 y - \omega_2 x)$

$$\begin{aligned}\mathrm{div}\boldsymbol{v} &= \frac{\partial}{\partial x}(\omega_2 z - \omega_3 y) + \frac{\partial}{\partial y}(\omega_3 x - \omega_1 z) + \frac{\partial}{\partial z}(\omega_1 y - \omega_2 x)\\ &= 0 + 0 + 0 = 0\end{aligned}$$

< 48 ページ. ハミルトンの演算子 >

問の解答

$$
\begin{aligned}
\nabla \cdot (\nabla \times \boldsymbol{F}) &= \left(\frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\,,\ \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\,,\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\,,\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) \\
&= \frac{\partial^2 f_3}{\partial y \partial x} - \frac{\partial^2 f_2}{\partial z \partial x} + \frac{\partial^2 f_1}{\partial z \partial y} - \frac{\partial^2 f_3}{\partial x \partial y} + \frac{\partial^2 f_2}{\partial x \partial z} - \frac{\partial^2 f_1}{\partial y \partial z} \\
&= \left(\frac{\partial^2 f_3}{\partial y \partial x} - \frac{\partial^2 f_3}{\partial x \partial y}\right) + \left(-\frac{\partial^2 f_2}{\partial z \partial x} + \frac{\partial^2 f_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 f_1}{\partial z \partial y} - \frac{\partial^2 f_1}{\partial y \partial z}\right) \\
&= 0 + 0 + 0 = 0
\end{aligned}
$$

< 49 ページ. ポテンシャル >

問 1 の解答

(1) $\boldsymbol{v} = (x - b_1\ ,\ y - b_2\ ,\ z - b_3)$

$$U(x, y, z) = -\frac{1}{2}(x - b_1)^2 - \frac{1}{2}(y - b_2)^2 - \frac{1}{2}(z - b_3)^2$$

(2) $\boldsymbol{v} = (-x\ ,\ -y\ ,\ 2z)$

$$U(x, y, z) = \frac{x^2}{2} + \frac{y^2}{2} - z^2$$

問 2 の解答

$$\varphi(x, y, z) = \int_a^x f_1(t, y, z)dt + \int_b^y f_2(a, t, z) + \int_c^z f_3(a, b, t)dt$$

$$\frac{\partial \varphi}{\partial x} = f_1(x, y, z)$$

$$\begin{aligned}
\frac{\partial \varphi}{\partial y} &= \int_a^x \frac{\partial f_1}{\partial y}(t, y, z)dt + f_2(a, y, z) \\
&= \int_a^x \frac{\partial f_2}{\partial x}(t, y, z)dt + f_2(a, y, z) \\
&= \Big[f_2(t, y, z)\Big]_{t=a}^{t=x} + f_2(a, y, z) \\
&= f_2(x, y, z) - f_2(a, y, z) + f_2(a, y, z) \\
&= f_2(x, y, z)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial y}{\partial z} &= \int_a^x \frac{\partial f_1}{\partial z}(t, y, z)dt + \int_b^y \frac{\partial f_2}{\partial z}(a, y, z)dt + f_3(a, y, z) \\
&= \int_a^x \frac{\partial f_3}{\partial x}(t, y, z)dt + \int_b^y \frac{\partial f_3}{\partial y}(a, t, z)dt + f_3(a, y, z) \\
&= \Big[f_3(t, y, z)\Big]_{t=a}^{t=x} + \Big[f_3(a, t, z)\Big]_{t=b}^{t=y} + f_3(a, b, z) \\
&= f_3(x, y, z) - f_3(a, y, z) + f_3(a, y, z) - f_3(a, b, z) + f_3(a, b, z) \\
&= f_3(x, y, z)
\end{aligned}$$

よって

$$\begin{aligned}
\nabla \varphi(x, y, z) = \left(\frac{\partial \varphi}{\partial x}\ ,\ \frac{\partial \varphi}{\partial y}\ ,\ \frac{\partial \varphi}{\partial z}\right) &= \Big(f_1(x, y, z)\ ,\ f_2(x, y, z)\ ,\ f_3(x, y, z)\Big) \\
&= \boldsymbol{F}(x, y, z)
\end{aligned}$$

より $\qquad \nabla \varphi = \boldsymbol{F}$

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問の解答

$$\int_C \nabla\varphi \cdot d\boldsymbol{r} = \int_S \mathrm{rot}(\nabla\varphi) \cdot d\boldsymbol{S} = \int_S \nabla \times (\nabla\varphi) \cdot d\boldsymbol{S} = \int_S \mathbf{0} \cdot d\boldsymbol{S} = 0$$