LETTER *Special Section on Information Theory and Its Applications*

ISI-Free Power Roll-Off Pulse**

Masayuki MOHRI[†*], ***Nonmember and*** **Masanori HAMAMURA**[††a)], ***Member***

SUMMARY An ISI-free power roll-off pulse, the roll-off characteristic of which is tunable with one power parameter, is proposed. It is shown that the proposed pulse is advantageous in terms of the probability of error for pulse detection in the presence of a timing error among currently known good pulses, among which the raised cosine pulse, "better than" raised cosine pulse, and polynomial pulse are considered.
key words: *intersymbol interference, pulse analysis, pulse shaping, timing error*

1. Introduction

The requirement for pulses to be free of intersymbol interference (ISI) is Nyquist's pulse-shaping criterion [1], [2], given by

$$\sum_{k=-\infty}^{\infty} S\left(f - \frac{k}{T}\right) = 1, \tag{1}$$

where $S(f)$ is the Fourier transform of the ISI-free pulse $s(t)$. A pulse that meets criterion (1) causes no ISI at equally spaced intervals that are integer-multiples of T[s]. One of the well-known pulse shapes that satisfy this criterion is the raised cosine (RC) pulse. Beaulieu, Tan, and Damen proposed the "better than" raised cosine (BTRC) pulse, which has a smaller maximum sidelobe than the RC pulse at the same excess bandwidth and results in a smaller probability of error in the presence of a detection timing error [3]. To further reduce sidelobe level, Sandeep, Chandan, and Chaturvedi proposed a linear combination of the RC and BTRC pulses [4]. This pulse has one design parameter. Recently, they have proposed a family of polynomial (POLY) pulses that can employ several design parameters [5], and have shown that, using three design parameters, the fourth-degree POLY pulses can outperform known good pulses in terms of the probability of error in the presence of a detection timing error. Although a family of POLY pulses is advantageous in designing pulse shape, we face a difficulty when the parameters must be optimized.

In this paper, we discuss a new ISI-free power roll-off (POWER) pulse, the shape of which is tunable with one parameter that is easily optimized, and show that this POWER pulse achieves a smaller probability of error than the fourth-degree POLY pulses investigated in [5] in most cases of the commonly considered benchmark test, with one exception.

2. Known ISI-Free Pulses

2.1 Raised Cosine Pulse

The most popular ISI-free pulse is the RC pulse. Its frequency spectrum is given by

$$S_{rc}(f) = \begin{cases} 1, & 0 \le |f| \le (1-\alpha)B \\ \dfrac{1}{2}\left\{1 + \cos\left(\dfrac{\pi}{2\alpha B}(|f| - (1-\alpha)B)\right)\right\}, & \\ & (1-\alpha)B \le |f| \le (1+\alpha)B \\ 0, & |f| > (1+\alpha)B \end{cases} \tag{2}$$

and its corresponding waveform is

$$s_{rc}(t) = \frac{\sin(2\pi Bt)}{\pi t} \cdot \frac{\cos(2\pi\alpha Bt)}{1-(4\alpha Bt)^2}, \tag{3}$$

where $B = 1/(2T)$ and the parameter α $(0 \le \alpha \le 1)$ is called the roll-off factor.

The bandwidth/pulse-rate trade-off depends on α. The bandwidth varies from a minimum of $W = 1/(2T)$ [Hz] ($\alpha = 0$) to a maximum of $W = 1/T$ [Hz] ($\alpha = 1$) [1]. In general, the tails of $s_{rc}(t)$ decay as $1/t^3$ for $\alpha > 0$.

2.2 "Better Than" Raised Cosine Pulse

Recently, a new pulse has been proposed in [3]. We refer to it as the BTRC pulse. Its frequency spectrum is given by

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[†]The author is with the Department of Information Systems Engineering, Kochi University of Technology, Kami-shi, 782-8502 Japan.
[††]The author is with the Graduate School of Engineering, Kochi University of Technology, Kami-shi, 782-8502 Japan.
[*]Presently, with Ryukoku University.
[**]This paper was partly presented at the 2008 International Symposium on Information Theory and its Applications (ISITA2008), Auckland, New Zealand, Dec. 2008.
a) E-mail: hamamura.masanori@kochi-tech.ac.jp
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$$S_{btrc}(f) = \begin{cases} 1, & 0 \le |f| \le (1-\alpha)B \\ \exp\left\{\dfrac{-\ln 2}{\alpha B}(|f| - (1-\alpha)B)\right\}, & \\ & (1-\alpha)B \le |f| \le B \\ 1 - \exp\left\{\dfrac{-\ln 2}{\alpha B}((1+\alpha)B - |f|)\right\}, & \\ & B \le |f| \le (1+\alpha)B \\ 0, & |f| > (1+\alpha)B \end{cases} \tag{4}$$

and its corresponding waveform is

$$s_{btrc}(t) = \frac{\sin(2\pi Bt)}{\pi t} \cdot \frac{4\pi\gamma t \sin(2\pi\alpha Bt) + 2\gamma^2 \cos(2\pi\alpha Bt) - \gamma^2}{4\pi^2 t^2 + \gamma^2}, \tag{5}$$

where $\gamma \stackrel{\Delta}{=} \ln 2/(\alpha B)$. In general, the tails of $s_{btrc}(t)$ decay as $1/t^2$ for $\alpha > 0$.

The BTRC pulse has a smaller maximum sidelobe than the RC pulse at the same excess bandwidth.

2.3 Linear Combination of Pulses

A linear combination (LC) of two ISI-free pulses yields a pulse that satisfies Nyquist's pulse-shaping criterion [4].

The LC of the RC and BTRC pulses (LC pulse) is given by

$$s_{lc}(t) = a\, s_{btrc}(t) + (1-a)s_{rc}(t). \tag{6}$$

2.4 Polynomial Pulse

A family of ISI-free polynomial (POLY) pulses were proposed [5]. Using three design parameters, the fourth-degree POLY pulse is given by

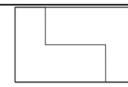
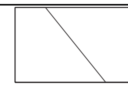
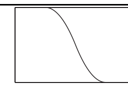
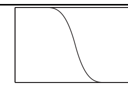
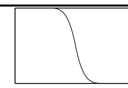
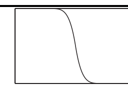
$$s_{poly}(t) = \frac{\sin(2\pi Bt)}{\pi t}\left\{\left(1 + \frac{a_2}{2} + \frac{a_3}{4} + \frac{a_4}{8}\right)\frac{\sin(2\pi\alpha Bt)}{2\pi\alpha Bt} - \frac{a_2}{2}\frac{\sin^2(\pi\alpha Bt)}{(\pi\alpha Bt)^2} + \frac{3a_3}{2}\frac{(\sin(2\pi\alpha Bt)/(2\pi\alpha Bt) - 1)}{(2\pi\alpha Bt)^2} + \frac{3a_4}{8}\frac{(\sin^2(\pi\alpha Bt)/(\pi\alpha Bt)^2 - 1)}{(\pi\alpha Bt)^2}\right\}. \tag{7}$$

3. Proposed Pulse

The frequency spectrum of the POWER pulse has a simple form raised to the β power; it is given by

$$S_p(f) = \begin{cases} 1, & 0 \le |f| \le (1-\alpha)B \\ 1 - \dfrac{1}{2}\left\{\dfrac{|f| - (1-\alpha)B}{\alpha B}\right\}^{\beta}, & (1-\alpha)B \le |f| \le B \\ \dfrac{1}{2}\left\{\dfrac{(1+\alpha)B - |f|}{\alpha B}\right\}^{\beta}, & B \le |f| \le (1+\alpha)B \\ 0, & |f| > (1+\alpha)B \end{cases} \tag{8}$$

Table 1 Special cases of the POWER pulse. ($b(t) = \sin(2\pi Bt)/(\pi t)$, $\nu = \pi\alpha B$)

β	Spectrum $S_p(f)$	Waveform Function $s_p(t)$
0		$b(t) \cdot \cos(2\nu t)$
1		$b(t) \cdot \dfrac{\sin(2\nu t)}{2\nu t}$
2		$b(t) \cdot \dfrac{1 - \cos(2\nu t)}{2\nu^2 t^2}$
3		$b(t) \cdot \dfrac{3(2\nu t - \sin(2\nu t))}{4\nu^3 t^3}$
4		$b(t) \cdot \dfrac{3(2\nu^2 t^2 - 1 + \cos(2\nu t))}{2\nu^4 t^4}$
5		$b(t) \cdot \dfrac{5(4\nu^3 t^3 - 6\nu t + 3\sin(2\nu t))}{4\nu^5 t^5}$

and its corresponding waveform is

$$s_p(t) = \frac{\sin(2\pi Bt)}{\pi t}\left\{1 - \frac{4\pi^2\alpha^2 B^2 t^2}{2 + 3\beta + \beta^2} \cdot {}_1F_2\left(1; \frac{1}{2}(3+\beta), \frac{1}{2}(4+\beta); -\pi^2\alpha^2 B^2 t^2\right)\right\}, \tag{9}$$

where $B = 1/(2T)$, α ($0 \le \alpha \le 1$) is the roll-off factor, β (≥ 0) is the roll-off tuning parameter that changes the POWER pulse into various shapes, and ${}_1F_2(x; y_1, y_2; z)$ is a hypergeometric function given by

$${}_1F_2(x; y_1, y_2; z) = \sum_{n=0}^{\infty} \frac{(x)_n}{(y_1)_n (y_2)_n} \cdot \frac{z^n}{n!}, \tag{10}$$

where $(u)_0 = 1$, and $(u)_n = u(u+1)(u+2)\cdots(u+n-1)$.

The pulse waveform $s_p(t)$ given by (9) reduces to a much simpler form if β is a small integer, as shown in Table 1. Note that for $\alpha > 0$, the tails of $s_p(t)$ decay as $1/t$ if $\beta = 0$, as $1/t^2$ if $\beta = 1$, and as $1/t^3$ if $\beta \ge 2$.

4. Error Probabilities

The probability of error for pulse detection in the presence of a timing error can be evaluated as [6]

$$P_e = \frac{1}{2} - \frac{2}{\pi}\sum_{\substack{m=1 \\ m \text{ odd}}}^{M} \left\{\frac{\exp(-m^2\omega^2/2)\sin(m\omega g_0)}{m}\right\} \prod_{\substack{k=N_1 \\ k \ne 0}}^{N_2} \cos(m\omega g_k), \tag{11}$$

where M represents the number of coefficients considered in the approximate Fourier series of the noise complementary

Table 2 Error probabilities for the POWER pulse with $N_1 = -512$, $N_2 = 512$, and SNR = 15 dB.

α	Pulse	t_e/T ±0.05	±0.10	±0.20
0.25	RC	8.219e-08	2.818e-06	9.746e-04
	BTRC	5.812e-08	1.298e-06	3.568e-04
	LC (a = 1.82)	5.119e-08	1.027e-06	2.697e-04
	POLY (a_2=40, a_3=−100, a_4=85)	4.734e-08	8.834e-07	2.241e-04
	POWER (β = 0.25)	4.576e-08	8.243e-07	2.048e-04
0.35	RC	6.000e-08	1.390e-06	3.908e-04
	BTRC	3.925e-08	5.402e-07	1.013e-04
	LC (a = 1.59)	3.503e-08	4.456e-07	8.283e-05
	POLY (a_2=31, a_3=−80, a_4=69)	3.290e-08	3.839e-07	6.563e-05
	POWER (β = 0.33)	3.103e-08	3.564e-07	6.434e-05
0.50	RC	3.972e-08	5.489e-07	1.022e-04
	BTRC	2.413e-08	1.858e-07	2.088e-05
	LC (a = 1.41)	2.208e-08	1.612e-07	1.962e-05
	POLY (a_2=25, a_3=−64, a_4=55)	2.057e-08	1.354e-07	1.520e-05
	POWER (β = 0.37)	1.945e-08	1.250e-07	1.662e-05
1.00	RC	1.528e-08	5.872e-08	3.654e-06
	BTRC	1.315e-08	3.569e-08	1.614e-06
	LC (a = 0.69)	1.229e-08	2.942e-08	1.362e-06
	POLY (a_2, a_3, a_4 : Unknown)	-	-	-
	POWER (β = 1.02)	1.228e-08	2.941e-08	1.359e-06

distribution function, ω is defined as $\omega = 2\pi/T_a$ using the period T_a in the series, N_1 and N_2 represent the numbers of interfering pulses before and after the pulse of interest, respectively, and g_k is the sampled value of the pulse, $s(t-kT)$ at $t = t_e$, that is, $g_k = s(t-kT)|_{t=t_e}$.

We commonly consider $N_1 = -512$, $N_2 = 512$, and a signal-to-noise ratio (SNR) of 15 dB for a benchmark test, and as was mentioned in [6], the error probabilities are calculated using $T_a = 30$ and $M = 31$. We numerically choose optimal β values that minimize the probability of error for $t_e = \pm 0.1T$. Note that the chosen β values are optimal for only $t_e = \pm 0.1T$, not for $t_e = \pm 0.05T$ and $\pm 0.20T$. The probabilities of error for the RC, BTRC, LC, POLY, and POWER pulses are listed in Table 2 for typical values of roll-off factor α. The POWER pulse achieves the smallest error probability for all t_e and α values, except in one case of $\alpha = 0.50$ and $t_e = \pm 0.20T$, in which $P_e = 1.520 \times 10^{-5}$ for the POLY pulse and $P_e = 1.662 \times 10^{-5}$ for the POWER pulse.

5. Conclusion

An ISI-free pulse that has one shape-tuning parameter for given roll-off factors has been proposed, and the probabilities of error in the presence of a detection timing error have been discussed. It has been shown that the proposed pulse achieves the smallest error probability in most cases of commonly considered benchmark tests.

References

[1] H.P. Hsu, Analog and Digital Communications, 2nd ed., McGraw-Hill, New York, 2003.

[2] J.G. Proakis, Digital Communications, 4th ed., McGraw-Hill, New York, 2000.

[3] N.C. Beaulieu, C.C. Tan, and M.O. Damen, "A "better than" Nyquist pulse," IEEE Commun. Lett., vol.5, no.9, pp.367–368, Sept. 2001.

[4] P. Sandeep, S. Chandan, and A.K. Chaturvedi, "ISI-free pulses with reduced sensitivity to timing errors," IEEE Commun. Lett., vol.9, no.4, pp.292–294, April 2005.

[5] S. Chandan, P. Sandeep, and A.K. Chaturvedi, "A family of ISI-free polynomial pulses," IEEE Commun. Lett., vol.9, no.6, pp.496–498, June 2005.

[6] N.C. Beaulieu, "The evaluation of error probabilities for intersymbol and cochannel interference," IEEE Trans. Commun., vol.39, no.12, pp.1740–1749, Dec. 1991.